LECTURE 10: SECOND ORDER DIFFERENTIAL EQUATIONS

PS: Next week: Lecture is Tuesday

Review: First order exact ODE's

Example: $1 + 3x^2 \cdot x' = 0$

Solution: $1 + 3x^2 \cdot x' = 0$

Try to rewrite the equation in this form

$$\frac{d}{dt}\left(u(x,t)\right) = 0 \iff \text{Solution: } \boxed{u(x,t) = C}$$

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} \cdot x' = 0 \iff \frac{\partial u}{\partial t} = 1 \qquad \frac{\partial u}{\partial x} = 3x^2$$

Is this possible: $\frac{\partial}{\partial x}(1) = 0 \qquad \frac{\partial}{\partial t}(3x^2) = 0$

equality = exact ODE

Find u: $\frac{\partial u}{\partial t} = 1 \Rightarrow u = t + c(x)$

$$\frac{\partial u}{\partial x} = 0 + c'(x) = 3x^2 \Rightarrow c(x) = x^3 \Rightarrow u = t + x^3$$

General solution: $u(x,t) = C \qquad t + x^3 = C \qquad x^3 = C - t$

$$x = \sqrt[3]{C - t}$$

Definition: A second order differential equation can be written as

$$\boxed{y'' = F(y', y, t)}$$

where F is some function, $y'' = y''_{tt}$, $y' = y'_t$.

Examples:

a) $y'' = 1$

b) $y'' = 3t - t^3$

c) $y'' - y = 4 \iff y'' = y + 4$

d) $y'' - 2y' + y = 0 \iff y'' = 2y' - y$

e) $y'' - 4y' + 3y = te^t \iff y'' = 4y' - 3y + te^t$

examples of second order ODE's

Example: Solve the second order ODE's

a) $x'' = 1$

b) $y'' = y' + t$

Recall:
A differential equation of order n has a general solution that depends on n parameters

Solution to a):

$x'' = 1$

$x' = \int 1\, dt = t + C_1$

$x' = t + C_1$

$x = \int (t + C_1)\, dt = \frac{1}{2}t^2 + C_1 t + C_2$

General: $x = \frac{1}{2}t^2 + C_1 t + C_2$

Solution to b):

$y'' = y' + t \xrightarrow{u = y'}$

$u' = u + t$

$u' - u = t \quad IF = e^{-t}$

$(u \cdot e^{-t})' = te^{-t}$

$u \cdot e^{-t} = \int te^{-t} dt = t \cdot (-e^{-t}) - \int 1 \cdot (-e^{-t}) dt$

$u \cdot e^{-t} = -te^{-t} + (-e^{-t}) + C_1$

$u \cdot e^{-t} = -te^{-t} - e^{-t} + C_1$

$u = -t - 1 + C_1 e^{t}$

$y' = u$

$y' = -t - 1 + C_1 e^{t}$

$y = \int (-t - 1 + C_1 e^{t})\, dt = -\frac{1}{2}t^2 - t + C_1 e^{t} + C_2$

Linear second order differential equations:

Defn: $y'' + a(t)\cdot y' + b(t)\cdot y = c(t)$

$y'' + ay' + by = c(t)$

Constant coefficients: $a(t) = a$, $b(t) = b$ are constants

Homogeneous: $c(t) = 0$

Examples:

a) $y'' - 4y' + 3y = 0$ linear, second order, homogeneous differential equation with constant coefficients

b) $y'' - t\cdot y' + e^t y = 3t$ linear second order ODE

c) $y'' - 7y' + 12y = te^t$ linear second order ODE

Compare: Linear first order ODE's

$y' + a(t)y = b(t)$

$y' + ay = b(t)$ $\quad (ye^{at})' = b(t)e^{at}$

$ye^{at} = \int b(t)e^{at}\,dt$

$y = e^{-at}\int b(t)e^{at}\,dt$

$b(t) = b$ const:

$y = \frac{b}{a} + C\cdot e^{-at}$

Basic example:

Homogeneous
Const. coefficients
Different roots of char. eq.

$y'' - 7y' + 12y = 0 \qquad (*)$

i) Is e^{rt} a solution for some r?

Try e^{rt}: $\quad y = e^{rt} \quad y' = r\cdot e^{rt} \quad y'' = r^2\cdot e^{rt}$

$$y'' - 7y' + 12y = \boxed{r^2e^{rt} - 7\cdot re^{rt} + 12e^{rt} = 0}$$

$$e^{rt}(r^2 - 7r + 12) = 0$$

Characteristic equation: $\longrightarrow r^2 - 7r + 12 = 0$

$r = 3,\ r = 4$

Conclusion: e^{3t}, e^{4t} are solutions

ii) Fact: (for linear homog. eq.) If y_1 and y_2 are solutions of $(*)$, then $C_1\cdot y_1 + C_2\cdot y_2$ is also a solution of $(*)$.

Conclusion: $y = C_1\cdot e^{3t} + C_2e^{4t}$ is a solution of $(*)$.

Compare:

$y' - 2y = 0$

Solution:

$y = C_1e^{2t}$

Char. eq: $r - 2 = 0$

$r = 2$

iii) If e^{3t} and e^{4t} are "different", then

$y = C_1e^{3t} + C_2e^{4t}$

is the general solution of $(*)$.

Conclusion:

If $y''+ay'+by=0$ has two different characteristic roots $r_1 \neq r_2$, then the general solution is

$$y = C_1 \cdot e^{r_1 t} + C_2 e^{r_2 t}$$

Characterie equation:

$$r^2 + ar + b = 0$$

Characteristic roots:

solutions of char. eqn.

Example: $y'' - 4y' + 3y = 0$

Char. eq: $r^2 - 4r + 3 = 0$

$r=1,\ r=3$

$$y = C_1 e^{1\cdot t} + C_2 e^{3t}$$

$$= C_1 e^{t} + C_2 e^{3t}$$ general solution

Example: $y'' - 5y' + 6y = 0$, $y(1)=0$, $y'(1)=1$

Char. eqn: $r^2 - 5r + 6 = 0$

$r=2,\ r=3$ $\Rightarrow$ General solution: $y = C_1 e^{2t} + C_2 e^{3t}$

$y(1)=0$: $0 = C_1 e^2 + C_2 \cdot e^3 \rightarrow e^2 \cdot C_1 + e^3 \cdot C_2 = 0$ (1)

$y'(1) = 1$: $1 = 2C_1 e^2 + 3C_2 e^3$ $\qquad 2e^2 \cdot C_1 + 3e^3 \cdot C_2 = 1$ (2)

$y' = 2C_1 e^{2t} + 3C_2 e^{3t}$

$2\cdot(1) - (2)$: $(2e^3 - 3e^3)\cdot C_2 = -1$

$$C_2 = \frac{1}{e^3} = e^{-3}$$

(1): $e^2 \cdot C_1 + \underbrace{e^3 \cdot e^{-3}}_{1} = 0$

$$C_1 = \frac{-1}{e^2} = -e^{-2}$$

$C_1 = -e^{-2}$, $C_2 = e^{-3}$

Solution: $y = -e^{-2} \cdot e^{2t} + e^{-3} \cdot e^{3t}$

$$= e^{3t-3} - e^{2t-2}$$

Complication: When there is not two different characteristic roots.

Example: $y'' - 4y' + 4y = 0$

Char. eqn: $r^2 - 4r + 4 = 0$

$(r-2)^2 = 0$

One solution: $r = 2$

e^{2t} one solution

Try: $y = te^{rt} = t \cdot e^{2t}$

$y = te^{2t}$

$y' = 1 \cdot e^{2t} + t \cdot 2 \cdot e^{2t} = (1+2t)e^{2t}$

$y'' = 2 \cdot e^{2t} + (1+2t) \cdot 2 \cdot e^{2t}$

$= (4t+4)e^{2t}$

$y'' - 4y' + 4y = (4t+4)e^{2t} - 4(1+2t)e^{2t} + 4 \cdot te^{2t} = e^{2t}(4t+4-4-8t+4t)$

$= e^{2t} \cdot 0 = 0$

When r is a double root, then e^{rt} and $t \cdot e^{rt}$ are solutions. General solution: $y = C_1 \cdot e^{rt} + C_2 \cdot te^{rt}$

$$\left.\begin{array}{c} y'' - 4y' + 4y = 0 \\ r^2 - 4r + 4 = 0 \\ r = 2 \end{array}\right\} \quad \text{General solution: } y = C_1 \cdot e^{2t} + C_2 \cdot t \cdot e^{2t}$$

Linear second-order homogeneous ODE's with constant coeff.

$$y'' + ay' + by = 0 \qquad (a, b \text{ constants})$$

Characteristic equation: $r^2 + ar + b = 0$ $\longrightarrow$ Characteristic roots: $r = \dfrac{-a \pm \sqrt{a^2 - 4b}}{2}$

Cases:

General solution:

(1) Two different characteristic roots $r_1 \neq r_2$: $(a^2 - 4b > 0)$

$y = C_1 \cdot e^{r_1 t} + C_2 \cdot e^{r_2 t}$

(2) One (double) characteric root r: $(a^2 - 4b = 0)$

$y = C_1 \cdot e^{rt} + C_2 \cdot te^{rt}$

(3) No (real) characteristic roots: $(a^2 - 4b < 0)$

$\alpha = -\frac{a}{2} \quad \beta = \sqrt{b - \left(\frac{a}{2}\right)^2}$

$y = e^{\alpha t} \cdot \left(C_1 \cdot \cos(\beta t) + C_2 \sin(\beta t)\right)$

$= C_1 \cdot e^{\alpha t} \cos(\beta t) + C_2 \cdot e^{\alpha t} \sin(\beta t)$

Example: $y'' + 4y' + 7y = 0$

$$r^2 + 4r + 7 = 0 \quad \rightarrow \quad r = \frac{-4 \pm \sqrt{16-28}}{2} = -2 \pm \frac{\sqrt{-12}}{2} = -2 \pm \sqrt{-3}$$

$$\alpha = -\frac{a}{2} = \underline{-2} \qquad \beta = \sqrt{b - (a/2)^2} = \sqrt{7 - 2^2} = \underline{\sqrt{3}}$$

$$= \underset{\alpha}{\underset{\shortparallel}{-2}} \pm \underset{\beta}{\underset{\shortparallel}{\sqrt{3}}} \cdot \sqrt{-1}$$

General solution:

$$\underline{y = e^{-2t} \cdot \left(C_1 \cdot \cos \sqrt{3}t + C_2 \cdot \sin \sqrt{3}t \right)}$$

Example: $y'' - 7y' + 12y = 2t + 3$ — inhomogeneous linear second order ODE

homogenization

$y'' - 7y' + 12y = 0$: Solve homogeneous ODE:

$r^2 - 7r + 12 = 0$

$r = 3, r = 4$

$$y_h = C_1 e^{3t} + C_2 e^{4t}$$

y_h : general solution of the homogeneous ODE

Fact: The general solution of an inhomogeneous linear differential equation can be written in the form

$$\boxed{y = y_p + y_h}$$

y_p: any particular solution of the inhomogeneous eqn.

y_h : general solution of the homog. equation.

Example: $y'' - 7y' + 12y = 2t+3$

General solution:

$$y = y_p + \underbrace{C_1 e^{3t} + C_2 e^{4t}}_{y_h}$$

How can we find y_p? Variation of coefficients

Start with $\boxed{\begin{aligned} c(t) &= 2t+3 \\ c'(t) &= 2 \\ c''(t) &= 0 \end{aligned}}$ $\longrightarrow$ Guess: $y_p = A\cdot t + B$

$$y = \underbrace{\tfrac{1}{6}t + \tfrac{25}{72}}_{y_p} + \underbrace{C_1 e^{3t} + C_2 e^{4t}}_{y_h}$$

$$\left.\begin{aligned} y &= At+B \\ y' &= A \\ y'' &= 0 \end{aligned}\right\} \quad \begin{aligned} y'' - 7y' + 12y &= 0 - 7A + 12(At+B) \\ &= -7A + 12A\cdot t + 12B \\ &= (12A)\cdot t + (12B - 7A) = 2t+3 \end{aligned}$$

$12A = 2 \Rightarrow A = \frac{2}{12} = \frac{1}{6}$

$12B - 7A = 3 \Rightarrow 12B - \frac{7}{6} = 3$

$12B = 3 + \frac{7}{6} = \frac{18+7}{6} = \frac{25}{6}$

$B = \frac{25}{6\cdot 12} = \frac{25}{72}$

$$\begin{aligned} y_p &= At + B \\ &= \tfrac{1}{6}t + \tfrac{25}{72} \end{aligned}$$

Example: $y'' - 4y' + 4y = t^2 + 2$ $\qquad y = y_p + y_h$

Find y_h: $y'' - 4y' + 4y = 0$

$r^2 - 4r + 4 = 0$

$r = 2 \longrightarrow y_h = C_1\cdot e^{2t} + C_2\cdot te^{2t}$

Find y_p:

$$\left.\begin{aligned} c(t) &= t^2+2 \\ c'(t) &= 2t \\ c''(t) &= 2 \end{aligned}\right\} \quad \begin{aligned} y &= At^2 + Bt + C \\ y' &= 2At + B \\ y'' &= 2A \end{aligned}$$

$(2A) - 4(2At + B) + 4(At^2 + Bt + C) = t^2 + 2$

$(4A)\cdot t^2 + (4B - 8A)t + (2A - 4B + 4C) = t^2 + 2$

$4A = 1 \Rightarrow A = 1/4$

$4B - 8A = 0$

$4B - 2 = 0 \quad B = \frac{1}{2}$

$2A - 4B + 4C = 2$

$2\cdot\frac{1}{4} - 4\cdot\frac{1}{2} + 4C = 2$

$4C = 2 + 2 - \frac{1}{2} = \frac{7}{2}$

$C = \frac{7}{8}$

$y_p = \frac{1}{4}t^2 + \frac{1}{2}t + \frac{7}{8}$

$$y = \underbrace{\tfrac{1}{4}t^2 + \tfrac{1}{2}t + \tfrac{7}{8}}_{y_p} + \underbrace{C_1 e^{2t} + C_2 te^{2t}}_{y_h}$$ general solution

Example: $y'' - 7y' + 12y = t \cdot e^{2t}$

How to find y_p:

$$\left.\begin{aligned} c(t) &= te^{2t} \\ c'(t) &= 1\cdot e^{2t} + t\cdot 2e^{2t} = (1+2t)e^{2t} \\ c''(t) &= 2e^{2t} + (1+2t)2\cdot e^{2t} = (4t+4)e^{2t} \end{aligned}\right\}$$

Guess: $y = \underline{(At+B)e^{2t}}$

Comment: In some special cases, the initial guess for y_p might not work (i.e. no solutions for $A, B, \ldots$)
In that case, you can try to multiply the initial guess with t and try again.