0. Review of matrix algebra and determinants

Reading. This lecture covers topics from Section 1.1, pages 2-4 in FMEA [**2**]. For a more detailed treatment, see Sections 15.2-15.5 and 16.1-16.5 in EMEA [**3**]. See also 8.1-8.3 in [**1**]. Note that most of this material is part of the required background for the course.

0.1. Matrix algebra. In this section we review the notion of matrices, matrix summation and matrix multiplication.

Matrix addition and subtraction. We first look at the notion of a matrix.

Definition 0.1. A matrix is a rectangular array of numbers considered as an entity.

We write down two matrices of small size.

Example 0.2.

$$A = \begin{pmatrix} 2 & 1 \\ 3 & 6 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & 2 \\ 1 & 0 \\ 6 & 5 \end{pmatrix}$$

Here A is a 2×2 matrix (two by two matrix) and B is a 3×2 matrix. We also say that A has size, order or dimension 2×2.

An $m \times n$ matrix is a matrix with m rows and n columns. If we denote the matrix by A, the number at the ith row and at the jth column is often denoted by a_{ij} :

$$A = \left.\underbrace{\begin{pmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{pmatrix}}_{n \text{ columns}}\right\} m \text{ rows}$$

am1

diagonal

We often write $A = (a_{ij})_{m \times n}$ for short. The *diagonal* entries of A are a_{11}, a_{22}, $a_{33}, \ldots$. In the case $m = n$, these elements form the *main diagonal*. A *zero matrix* is matrix consisting only of 0. A zero matrix is often written 0. The size of a zero matrix is usually clear from the context.

We say that two matrices are *equal* if the have they same size (order) and their corresponding entries are equal.

A word on brackets: A matrix is usually embraced by a pair of brackets, () or []. The book FMEA [**2**] uses (), so I try to use these pairs of brackets in these notes. During the lectures I will however mostly write [], since it is easier.

Definition 0.3. The sum of to matrices of same the order is computed by adding the corresponding entries.

It is easy to see how this works in an example.

2×2 $\quad$ 3×2 $\quad$ 2×2

Example 0.4.

$$A=\begin{pmatrix} 2 & 1 \\ 3 & 6 \end{pmatrix} \qquad B=\begin{pmatrix} 3 & 2 \\ 1 & 0 \\ 6 & 5 \end{pmatrix} \qquad C=\begin{pmatrix} 4 & -1 \\ 2 & 0 \end{pmatrix}$$

We can calculate the sum of A and C

$$A+C=\begin{pmatrix} 2 & 1 \\ 3 & 6 \end{pmatrix}+\begin{pmatrix} 4 & -1 \\ 2 & 0 \end{pmatrix}=\begin{pmatrix} 2+4 & 1+(-1) \\ 3+2 & 6+0 \end{pmatrix}=\begin{pmatrix} 6 & 0 \\ 5 & 6 \end{pmatrix},$$

but the sum of A and B is not defined, since they do not have the same order.

A matrix can be multiplied by a number.

Definition 0.5. Let A be a matrix and let k be a real number. Then kA is calculated by multiplying each entry of A by k.

In an example this looks as follows:

Example 0.6. If

$$A=\begin{pmatrix} 2 & 1 \\ 3 & 6 \end{pmatrix}$$

we get that

$$4A=\begin{pmatrix} 4\cdot 2 & 4\cdot 1 \\ 4\cdot 3 & 4\cdot 6 \end{pmatrix}=\begin{pmatrix} 8 & 4 \\ 12 & 24 \end{pmatrix}.$$

If A and B are two matrices of the same size we define $A-B$ to be $A+(-1)B$. This means that matrices are subtracted by subtracting the corresponding entries. The following rules applies:

Proposition 0.7. Let A, B and C be matrices of the same size (order), and let r and s be scalars (numbers). Then:

(1) $A+B=B+A$
(2) $(A+B)+C=A+(B+C)$
(3) $A+0=A$
(4) $r(A+B)=rA+rB$
(5) $(r+s)A=rA+\cancel{r}A \leftarrow s$
(6) $r(sA)=(rs)A$

Matrix multiplication. In this section we review matrix multiplication which is a bit more complicated than addition and subtraction of matrices.

\# col. of A = \# rows of B

Definition 0.8. Let A be an $m \times n$ matrix and let B be an $n \times p$ matrix. The product AB is an $m \times p$ matrix that is calculated according to the following scheme:

$$
\begin{array}{c|c}
 & \begin{pmatrix} b_{11} & \dots & \mathbf{b}_{1j} & \dots & b_{1p} \\ \vdots & & \vdots & & \vdots \\ b_{n1} & \dots & \mathbf{b}_{nj} & \dots & b_{np} \end{pmatrix} = B \\
\hline
A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ \mathbf{a}_{i1} & \dots & \mathbf{a}_{in} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} & \begin{pmatrix} c_{11} & \dots & c_{1j} & \dots & c_{1p} \\ \vdots & & \vdots & & \vdots \\ c_{i1} & \dots & \mathbf{c}_{ij} & \dots & c_{ip} \\ \vdots & & \vdots & & \vdots \\ c_{m1} & \dots & c_{mj} & \dots & c_{mp} \end{pmatrix} = AB
\end{array}
$$

m rows
p cols.
$(m \times p)$

The element c_{ij} is computed as

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj}$$

You should think of this in the following way: The line through row i in A and the line through a column j in B meet in the entry c_{ij} in the product. The entry c_{ij} is obtained by multiplying the first number in row i in A by the first number in the column j in B, the second number in row i of A with the second number in column j of B and so on, and then adding all these numbers.

We see that the product AB have the same number of rows as A and the same number of columns as B.

2×2 $\qquad$ 2×2 $\qquad$ 2×2

$A \cdot B$

Example 0.9. We take $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & 3 \\ 0 & 1 \end{pmatrix}$. Then we get

$$
\begin{array}{c|c}
 & \begin{pmatrix} 4 & 3 \\ 0 & 1 \end{pmatrix} = B \\
\hline
A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} & \begin{pmatrix} 1 \cdot 4 + 2 \cdot 0 & 1 \cdot 3 + 2 \cdot 1 \\ 3 \cdot 4 + 4 \cdot 0 & 3 \cdot 3 + 4 \cdot 1 \end{pmatrix} = A \cdot B
\end{array}
$$

Thus we get that

$$AB = \begin{pmatrix} 1 \cdot 4 + 2 \cdot 0 & 1 \cdot 3 + 2 \cdot 1 \\ 3 \cdot 4 + 4 \cdot 0 & 3 \cdot 3 + 4 \cdot 1 \end{pmatrix} = \begin{pmatrix} 4 & 5 \\ 12 & 13 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} 4 & 3 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 5 \\ 12 & 13 \end{pmatrix}$$

We need another example.

Example 0.10. Let $A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 0 \end{pmatrix}$. Calculate BA.

2×2 $\qquad$ 3×2

$A \cdot B$ = not defined

$B \cdot A$ = 3×2 matrix

3×2 $\quad$ 2×2

Solution.

$$\begin{array}{c|c} & \begin{pmatrix}1&2\\0&3\end{pmatrix} = A \\ \hline B = \begin{pmatrix}0&1\\1&0\\2&0\end{pmatrix} & \begin{pmatrix}0&3\\1&2\\2&4\end{pmatrix} = B\cdot A\end{array}$$

Thus we get that

$$BA = \begin{pmatrix}0&3\\1&2\\2&4\end{pmatrix}.$$

Note that the product AB is not defined.

Matrix multiplication follows rules that we know from multiplication of numbers.

Proposition 0.11. We have the following rules for matrix multiplication:

(1) $(AB)C = A(BC)$ (associative law)
(2) $A(B+C) = AB + AC$ (left distributive law)
(3) $(A+B)C = AC + BC$ (right distributive law)
(4) $k(AB) = (kA)B = A(kB)$ where k is a scalar.

But: $AB \neq BA$

The following easy example shows how to calculate with the distributive laws.

Example 0.12. Assume that A and B are square matrices of the same order. Show that

$$(A+B)(A-B) = A^2 - B^2$$

if and only if $AB = BA$.

$$= A\cdot A + B\cdot A - AB - B\cdot B = A^2 - B^2$$

only if $BA = AB$

Solution. We have that

$$(A+B)(A-B) = A(A-B) + B(A-B)$$

by the right distributive law. By applying the left distributive law two times, we get

$$A(A-B) + B(A-B) = A^2 - AB + BA - B^2.$$

This is equal to $A^2 - B^2$ if and only if $-AB + BA = 0$ which is the same as to say that $AB = BA$. (Note that we may move a summand from one side of the equation to the other side of the equation if we change the sign. Why?)

Problem 0.1. Simplify the expression where A, B and C are matrices.

$$A(3B - C) + (A - 2B)C + 2B(C + 2A)$$

0.2. Determinants. In this section we review how to compute determinants and cofactors.

For any $n\times n$-matrix A, you can compute $\det(A) = |A|$

Definition 0.13. Let A be a general 2×2 matrix,

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix}.$$

The determinant $|A|$ of A is defined by

$$\det(A) = |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}.$$

Let us compute the determinant in an example.

Example 0.14. Compute

$$\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} \text{ and } \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix}.$$

Solution.

$$\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} = 2 \cdot 4 - 3 \cdot 1 = 8 - 3 = 5.$$

$$\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = 1 \cdot 2 - 2 \cdot 1 = 2 - 2 = 0.$$

Cofactors are very important. They will allow us to calculate the determinant of larger matrices and to find the inverse matrix.

Signs:

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

Definition 0.15. Let A be an $n \times n$ matrix. The cofactor A_{ij} is $(-1)^{i+j}$ times the determinant obtained by deleting row i and column j in A. $= C_{ij}$

Example 0.16. Let

$$A = \begin{pmatrix} 1 & 3 & 5 \\ 0 & 1 & 4 \\ 2 & 0 & 1 \end{pmatrix}.$$

Compute the cofactor A_{23}.

Solution.

$$A_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} = (-1)(1 \cdot 0 - 2 \cdot 3) = 6$$

Using cofactors, we can now calculate the determinant of larger matrices.

Definition 0.17. Assume that

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

cofactor expansion along first row

is an $n \times n$ matrix. Then the determinant $|A|$ of A is given by

$$|A| = a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n}.$$

Moreover, this is called the *cofactor expansion of* A along the first row.

In the case A is a 3×3 matrix,

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

this gives the following formula for the determinant:

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

We use this formula to calculate the determinant in an example.

Example 0.18. Compute
$$\begin{vmatrix} 1 & 3 & 5 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{vmatrix}.$$

Solution.
$$\begin{aligned}\begin{vmatrix} 1 & 3 & 5 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{vmatrix} &= 1 \cdot \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} - 3 \cdot \begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} + 5 \cdot \begin{vmatrix} 0 & 1 \\ 2 & 0 \end{vmatrix} \\ &= 1 \cdot (1 \cdot 1 - 0 \cdot 1) - 3 \cdot (0 \cdot 1 - 2 \cdot 1) + 5 \cdot (0 \cdot 0 - 2 \cdot 1) \\ &= 1 - 3(-2) + 5(-2) \\ &= -3.\end{aligned}$$

Actually, the concept of cofactor expansion is more general. The determinant may be computed by cofactor expansion along any row or column. We will not state this in a formal theorem, but rather we show how this works in an example.

Example 0.19. Let
$$A = \begin{pmatrix} 1 & 3 & 5 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix}.$$
Compute $|A|$ by cofactor expansion along the second row.

Solution. We get
$$\begin{aligned}|A| &= 0 \cdot A_{21} + 1 \cdot A_{22} + 1 \cdot A_{23} \\ &= 0 \cdot (-1)^{2+1} \cdot \begin{vmatrix} 3 & 5 \\ 0 & 1 \end{vmatrix} + (-1)^{2+2} \cdot \begin{vmatrix} 1 & 5 \\ 2 & 1 \end{vmatrix} + (-1)^{2+3} \cdot \begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} \\ &= 0 + (1 \cdot 1 - 2 \cdot 5) - (1 \cdot 0 - 2 \cdot 3) \\ &= 0 - 9 + 6 \\ &= -3.\end{aligned}$$

An important notion is the transpose of a matrix A. This is written A^T. The books FMEA [2] and EMEA [3] use the notation A', but A^T is more common, so I will stick to that notation.

$A^T = A^t = A'$ (notation)

Definition 0.20. Let A be an $n \times m$ matrix. The transpose of A, denoted A^T, is the matrix obtained form A by interchanging the rows and columns in A.

For a 3×3 matrix, the transpose matrix is easy to write down.

Example 0.21. Let
$$A = \begin{pmatrix} 1 & 3 & 5 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix}.$$
Write down A^T.

Solution.
$$A^T = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 1 & 0 \\ 5 & 1 & 1 \end{pmatrix}. \qquad A^T = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 1 & 0 \\ 5 & 1 & 1 \end{pmatrix}$$

We now return to calculation of determinants, and note the following properties. These are very useful, and can simplify calculations tremendously.

Proposition 0.22. Let A and B be $n \times n$ matrices.

(1) $|A^T| = |A|$

(2) $|AB| = |A||B|$ $|A+B| \neq |A| + |B|$

(3) If two rows in A are interchanged, the sign of the determinant changes.

(4) If a row in A is multiplied by a constant k, the determinant is multiplied by k.

(5) If a multiple of one row is added to an other row, the determinant is unchanged.

We demonstrate the utility of this proposition in some examples.

Example 0.23. Let

$$A = \begin{pmatrix} -1 & 0 & -2 \\ 2 & 0 & 4 \\ 5 & 1 & 1 \end{pmatrix}.$$

Compute $|A|$.

Solution. Since the determinant is unchanged if we take two times the first row and add to the second, we obtain:

$$|A| = \begin{vmatrix} -1 & 0 & -2 \\ 2 & 0 & 4 \\ 5 & 1 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 0 & -2 \\ 0 & 0 & 0 \\ 5 & 1 & 1 \end{vmatrix}.$$

By taking cofactor expansion along the second row, we see that $|A| = 0$.

Example 0.24. We have that

$$\begin{vmatrix} 1 & 3 & 5 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{vmatrix} = -3.$$

Compute

$$\begin{vmatrix} -2 & -6 & -10 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{vmatrix} \text{ and } \begin{vmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 3 & 5 \end{vmatrix}.$$

Solution.

Note that $\begin{vmatrix} -2 & -6 & -10 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{vmatrix}$ is obtained from $\begin{vmatrix} 1 & 3 & 5 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{vmatrix}$ by multiplying the first row by -2. Thus

$$\begin{vmatrix} -2 & -6 & -10 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{vmatrix} = (-2)(-3) = 6.$$

Note further that $\begin{vmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 3 & 5 \end{vmatrix}$ is obtained from $\begin{vmatrix} 1 & 3 & 5 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{vmatrix}$ by interchanging two rows.

Thus

$$\begin{vmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 3 & 5 \end{vmatrix} = (-1)(-3) = 3.$$

Example 0.25. Let
$$A = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$
Compute $|AB|$.

$$|AB| = |A| \cdot |B| = (0 \cdot 1 - 0 \cdot 1) \cdot (\ldots) = 0$$

Solution.
$$|A| = 0 \implies |AB| = |A||B| = 0 \cdot |B| = 0.$$

Problem 0.2. Compute the determinants.

(a) $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{vmatrix}$

(b) $\begin{vmatrix} 1 & 2 & 3 \\ 1 & 4 & 3 \\ 1 & 2 & 6 \end{vmatrix}$

Defn: A is symmetric if $A^T = A$

Ex: $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 7 & 5 \end{pmatrix}$ $A^T = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 7 & 5 \end{pmatrix}$

$\Rightarrow$ A is symmetric

Fact: $(A^t)^t = A$

Before next lecture:

Exercise problem 0.1 – 0.12
(Exercise session Thursday 17–19)