Review: Kuhn-Tucker conditions

To review Kuhn-Tucker conditions, we shall start by solving the following optimization problem with inequality constraints:

$x^2+y-2 \le 0$ $\quad -y \le -1 \Leftrightarrow \boxed{-y+1 \le 0}$

Example

Maximize $f(x, y) = xy + x^2$ *subject to* $x^2 + y \le 2$ *and* $y \ge 1$.

Solution

The full solution will be given during Lecture 8. The short answer is that $f(1, 1) = 2$ *is the maximum.*

(a) There is a max since we can use EVT = Extreme Value Theorem
(b) ~~three~~ soln. to Kuhn-Tucker; max value: $f(1,1) = 2$
(c) ~~N~~DCQ not satisfied: no points where NDCQ not satisfied

Max: $x=1, y=1$, $f(1,1)=2$

Problem: Max $f(x,y) = xy + x^2$ subject to $\begin{cases} g_1(x,y) = x^2+y-2 \le 0 \\ g_2(x,y) = -y+1 \le 0 \end{cases}$

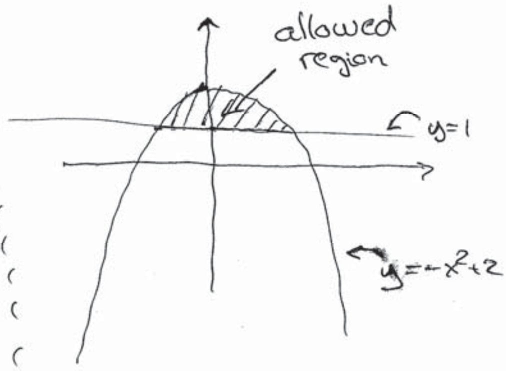
$\mathcal{L} = xy + x^2 - \lambda_1(x^2+y-2) - \lambda_2(-y+1)$

$\frac{\partial \mathcal{L}}{\partial x} = y + 2x - \lambda_1 \cdot 2x = 0$

$\frac{\partial \mathcal{L}}{\partial y} = x - \lambda_1 \cdot 1 - \lambda_2 \cdot (-1) = 0$

First order conditions:

$$\boxed{\begin{aligned} &y + 2x(1-\lambda_1) = 0 \\ &x - \lambda_1 + \lambda_2 = 0 \end{aligned}}$$



Complementary slackness condition:

$$\boxed{\begin{aligned} &\lambda_1 \ge 0, \quad \lambda_2 \ge 0 \\ &\lambda_1 = 0 \text{ if } x^2+y-2 < 0 \\ &\lambda_2 = 0 \text{ if } -y+1 < 0 \end{aligned}}$$

Constraints:

$$\boxed{\begin{aligned} x^2+y-2 &\le 0 \\ y &\ge 1 \end{aligned}}$$

Case I: $x^2+y-2=0$, $y=1$ $\qquad$ $g_1 = x^2+y-2 \le 0$, $g_2 = -y+1 \le 0$

$y=1$, $x^2=1 \Leftrightarrow x=\pm 1$

FOC: $y+2x(1-\lambda_1)=0$, $x-\lambda_1+\lambda_2=0$

$x=1$: $1+2(1-\lambda_1)=0$, $3-2\lambda_1=0$, $\lambda_1=3/2$; $1-3/2+\lambda_2=0$, $\lambda_2=1/2$

$x=-1$: $1-2(1-\lambda_1)=0$, $-1+2\lambda_1=0$, $\lambda_1=1/2$; $-1-1/2+\lambda_2=0$, $\lambda_2=3/2$

CSC: $\lambda_1 \ge 0$, $\lambda_2 \ge 0$

Solution in case I: $(x,y,\lambda_1,\lambda_2) = \begin{cases} (1,1,3/2,1/2) \\ (-1,1,1/2,3/2) \end{cases}$ $\qquad$ $f(1,1)=1+1=2$, $f(-1,1)=-1+1=0$

NDCQ: $\text{rk}\begin{pmatrix} 2x & 1 \\ 0 & -1 \end{pmatrix} = 2 \Leftrightarrow -2x \neq 0$

NDCQ not ~~qualified~~ satisfied: $-2x=0$, $x=0$ $\qquad$ $\left.\begin{array}{l} x=0 \\ y=1 \end{array}\right\}$ $x^2+y-2=-2\neq 0$

Cond: NDCQ ok for all points in case I.

Case 2: $x^2+y-2=0$, $y>1$

FOC: $y+2x(1-\lambda_1)=0$

$x-\lambda_1+\lambda_2=0 \Rightarrow x=\lambda_1 \Rightarrow y=2-x^2=2-\lambda_1^2$

CSC: $\lambda_1 \ge 0$, $\lambda_2 = 0$

$2-\lambda_1^2+2\lambda_1(1-\lambda_1)=0$

$2-\lambda_1^2+2\lambda_1-2\lambda_1^2=0$

$-3\lambda_1^2+2\lambda_1+2=0 \Rightarrow \lambda_1 = \dfrac{-2\pm\sqrt{4-4\cdot(-3)\cdot 2}}{-6}$

$= \dfrac{1}{3} \mp \dfrac{\sqrt{28}}{6} = \dfrac{1}{3} \mp \dfrac{\sqrt{7}}{3}$

$\lambda_1 = \dfrac{1+\sqrt{7}}{3}$, $\lambda_2=0$

$x=\dfrac{1+\sqrt{7}}{3}$, $y = 2-\dfrac{(1+\sqrt{7})^2}{9} = \dfrac{18-(1+2\sqrt{7}+7)}{9}$

$y<1 \Rightarrow$ no solution

Case 2: no solutions

NDCQ: $\text{rk}(2x \quad 1) = 1$ ok. $\Rightarrow$ NDCQ satisfied

Case 3: $x^2+y-2<0$, $\underline{y=1}$ $\qquad g_2 = -y+1 \leq 0$

FOC: $y + 2x(1-\lambda_1) = 0 \Rightarrow 1 + 2x \cdot 1 = 0 \Rightarrow x = \underline{-1/2}$

$x - \lambda_1 + \lambda_2 = 0 \Rightarrow -\frac{1}{2} - 0 + \lambda_2 = 0 \Rightarrow \underline{\lambda_2 = 1/2}$

CSC: $\underline{\lambda_1 = 0}$

$\lambda_2 \geq 0$

$x^2+y-2 = (-1/2)^2+1-2 = \frac{1}{4}-1 = -\frac{3}{4} < 0$ ok.

Solutions in case 3: $(x, y, \lambda_1, \lambda_2) = (-1/2, 1, 0, 1/2)$ $\qquad f(-1/2, 1) = -1/2 + 1/4 = \underline{-1/4}$

NDCQ: $\text{rk}\begin{pmatrix} 0 & -1 \end{pmatrix} = 1$ ok $\qquad$ NDCQ satisfied

Case 4: $x^2+y-2<0$, $y>1$

CSC: $\underline{\lambda_1 = 0}$

$\underline{\lambda_2 = 0}$

FOC: $y + 2x(1-\lambda_1) = 0$ $\quad$ $\underline{y=0}$

$x - \lambda_1 + \lambda_2 = 0$ $\quad$ $\underline{x=0}$

$y<1 \Rightarrow$ no solutions

Case 4: No solutions.

NDCQ satisfied