Section 1.1: Systems of Linear Equations

A **linear equation:**

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

{ $a_1, a_2, \ldots$; b — given numbers

EXAMPLE:

$4x_1 - 5x_2 + 2 = x_1$ and $x_2 = 2(\sqrt{6} - x_1) + x_3$

↓ rearranged ↓

$3x_1 - 5x_2 = -2$ and $2x_1 + x_2 - x_3 = 2\sqrt{6}$

Not linear:

$4x_1 - 6x_2 = x_1x_2$ and $x_2 = 2\sqrt{x_1} - 7$

A **system of linear equations** (or a **linear system**):

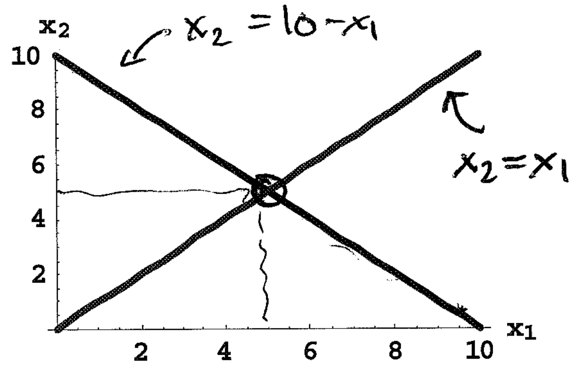
A collection of one or more linear equations involving the same set of variables, say, $x_1, x_2, \ldots, x_n$.

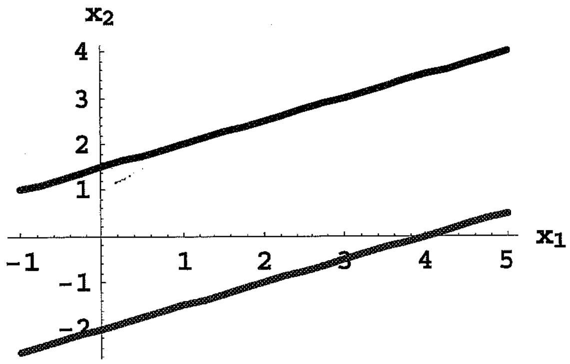
A **solution** of a linear system:

A list $(s_1, s_2, \ldots, s_n)$ of numbers that makes each equation in the system true when the values $s_1, s_2, \ldots, s_n$ are substituted for $x_1, x_2, \ldots, x_n$, respectively.

EXAMPLE Two equations in two variables:

$$\begin{aligned} x_1 + x_2 &= 10 \\ -x_1 + x_2 &= 0 \end{aligned} \qquad \begin{aligned} x_1 - 2x_2 &= -3 \\ 2x_1 - 4x_2 &= 8 \end{aligned}$$

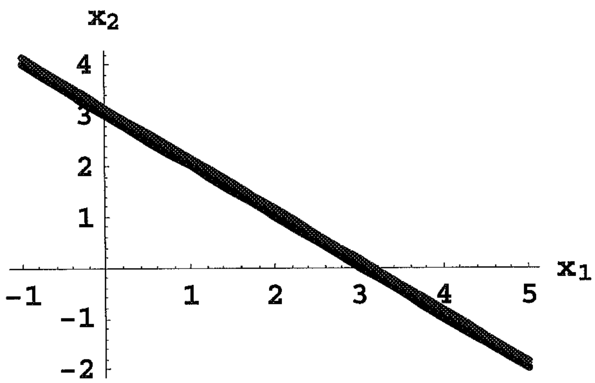




one unique solution **no solution**

$(x_1, x_2) = (5, 5)$

$$\begin{aligned} x_1 + x_2 &= 3 \\ -2x_1 - 2x_2 &= -6 \end{aligned}$$



infinitely many solutions

BASIC FACT: A system of linear equations has either

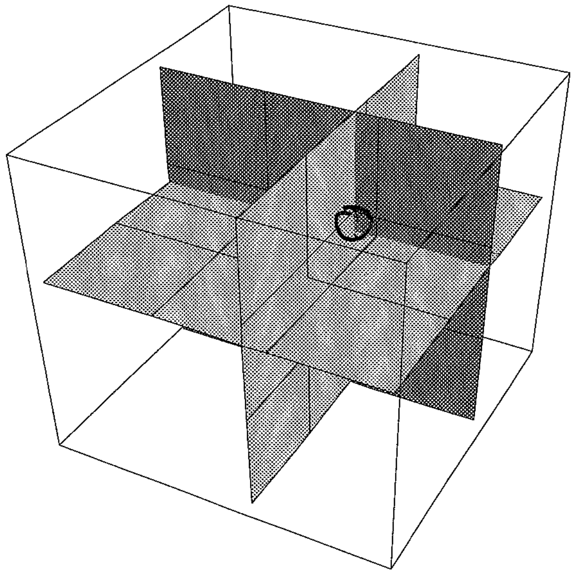
(i) exactly one solution *(consistent)* or

(ii) infinitely many solutions *(consistent)* or

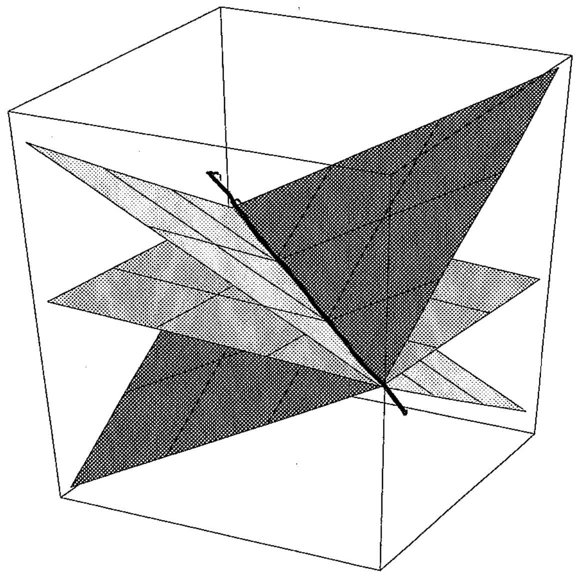
(iii) no solution *(inconsistent)*.

EXAMPLE: Three equations in three variables. Each equation determines a plane in 3-space.

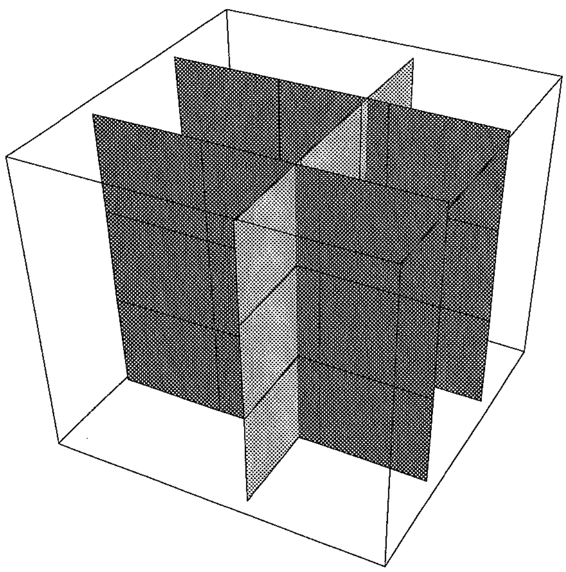
i) The planes intersect in one point. *(one solution)*



ii) The planes intersect in one line. *(infinitely many solutions)*



iii) There is not point in common to all three planes. *(no solution)*



Method: Gaussian elimination

The **solution set:**

- The set of all possible solutions of a linear system.

Equivalent systems:

- Two linear systems with the same solution set.

STRATEGY FOR SOLVING A SYSTEM:

- *Replace one system with an equivalent system that is easier to solve.*

EXAMPLE:

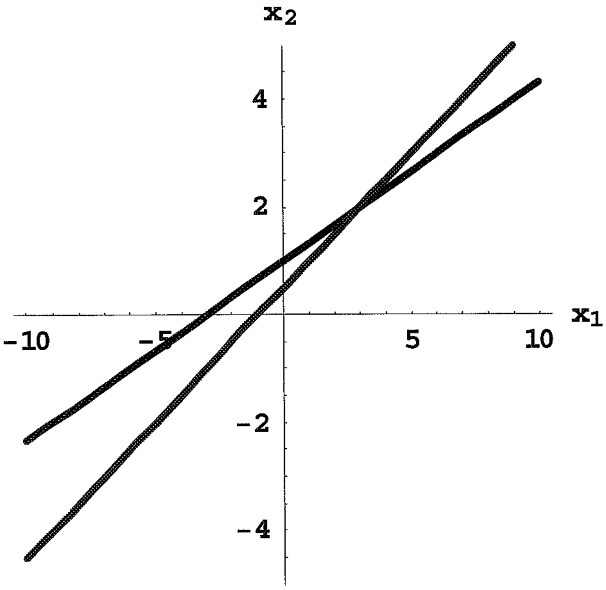
$$\begin{cases} (1) & x_1 - 2x_2 = -1 \\ (2) & -x_1 + 3x_2 = 3 \end{cases}$$

$\downarrow$

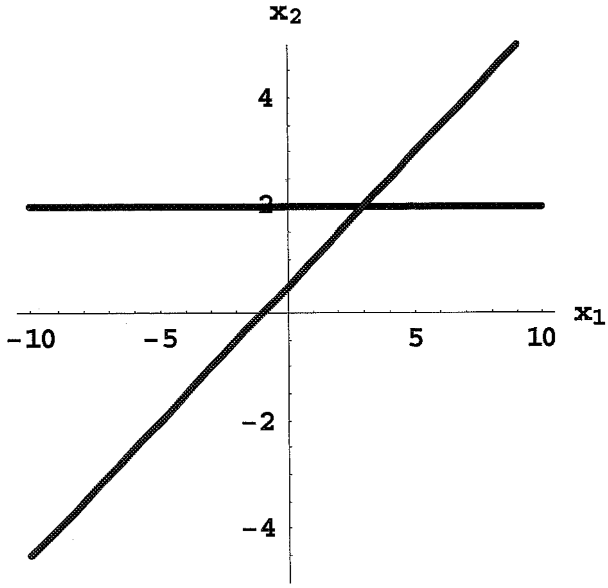
$$\begin{matrix} (i) = \\ (ii) = \end{matrix} \begin{cases} (1) & x_1 - 2x_2 = -1 \\ (1)+(2) & x_2 = 2 \end{cases}$$

$\downarrow$

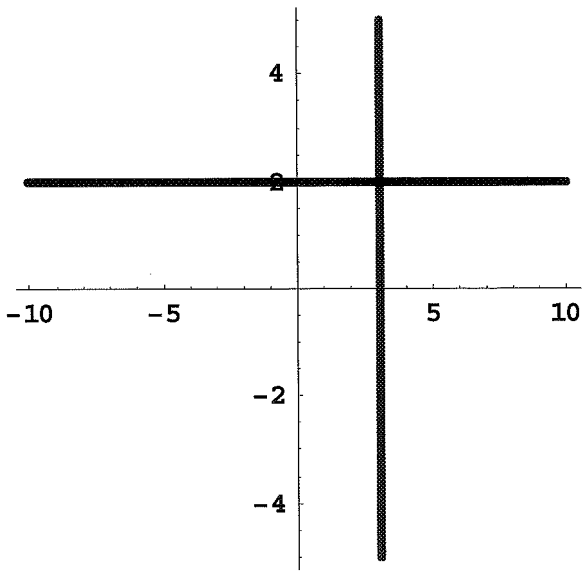
$$\begin{matrix} (i)+2(ii) \\ (ii) \end{matrix} \begin{cases} x_1 = 3 \\ x_2 = 2 \end{cases}$$



$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= 3 \end{aligned}$$



$$\begin{aligned} x_1 - 2x_2 &= -1 \\ x_2 &= 2 \end{aligned}$$



$$\begin{aligned} x_1 &= 3 \\ x_2 &= 2 \end{aligned}$$

Matrix Notation

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= 3 \end{aligned} \qquad \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$$

(coefficient matrix)

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= 3 \end{aligned} \qquad \left[\begin{array}{cc|c} 1 & -2 & -1 \\ -1 & 3 & 3 \end{array}\right]$$

(augmented matrix)

↓

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ x_2 &= 2 \end{aligned} \qquad \left[\begin{array}{cc|c} 1 & -2 & -1 \\ 0 & 1 & 2 \end{array}\right]$$

↓

$$\begin{aligned} x_1 &= 3 \\ x_2 &= 2 \end{aligned} \qquad \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \end{array}\right]$$

Elementary Row Operations:

1. *(Replacement)* Add one row to a multiple of another row.
2. *(Interchange)* Interchange two rows.
3. *(Scaling)* Multiply all entries in a row by a nonzero constant.

Row equivalent matrices: Two matrices where one matrix can be transformed into the other matrix by a sequence of elementary row operations.

Fact about Row Equivalence: If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.

3x3 linear system

EXAMPLE:

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right] \quad 4\cdot \quad (A|\underline{b})$$

augmented matrix

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ & - & 3x_2 & + & 13x_3 & = & -9 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right] \quad :2$$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & x_2 & - & 4x_3 & = & 4 \\ & - & 3x_2 & + & 13x_3 & = & -9 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array}\right] \quad 3\cdot$$

echelon form

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & x_2 & - & 4x_3 & = & 4 \\ & & & & x_3 & = & 3 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right] \quad 4\cdot \quad -1\cdot$$

$$\begin{array}{rcrcr} x_1 & - & 2x_2 & = & -3 \\ & & x_2 & = & 16 \\ & & x_3 & = & 3 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right] \quad 2$$

reduced echelon form

$$\begin{aligned} x_1 \quad\quad\quad &= 29 \\ x_2 \quad\quad &= 16 \\ x_3 &= 3 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Solution: $(29, 16, 3)$

Check: Is $(29, 16, 3)$ a solution of the *original* system?

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array}$$

$$\begin{array}{rcll} (29) - 2(16) + 3 & = 29 - 32 + 3 & = & 0 \\ 2(16) - 8(3) & = 32 - 24 & = & 8 \\ -4(29) + 5(16) + 9(3) & = -116 + 80 + 27 & = & -9 \end{array}$$

Two Fundamental Questions (Existence and Uniqueness)

1) Is the system consistent; (i.e. does a solution **exist**?)

2) If a solution exists, is it **unique**? (i.e. is there one & only one solution?)

EXAMPLE: Is this system consistent?

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array}$$

In the last example, this system was reduced to the triangular form:

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & x_2 & - & 4x_3 & = & 4 \\ & & & & x_3 & = & 3 \end{array} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

This is sufficient to see that the system is consistent and unique. Why?

EXAMPLE: Is this system consistent?

$$\begin{aligned} 3x_2 - 6x_3 &= 8 \\ x_1 - 2x_2 + 3x_3 &= -1 \\ 5x_1 - 7x_2 + 9x_3 &= 0 \end{aligned} \qquad \begin{bmatrix} 0 & 3 & -6 & 8 \\ 1 & -2 & 3 & -1 \\ 5 & -7 & 9 & 0 \end{bmatrix}$$

Solution: Row operations produce:

$$\begin{bmatrix} 0 & 3 & -6 & 8 \\ 1 & -2 & 3 & -1 \\ 5 & -7 & 9 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 3 & -1 \\ 0 & 3 & -6 & 8 \\ 0 & 3 & -6 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 3 & -1 \\ 0 & 3 & -6 & 8 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

Equation notation of triangular form:

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= -1 \\ 3x_2 - 6x_3 &= 8 \\ 0x_3 &= -3 \quad \leftarrow \textit{Never true} \end{aligned}$$

The original system is inconsistent!

EXAMPLE: For what values of h will the following system be consistent?

$$\begin{aligned} 3x_1 - 9x_2 &= 4 \\ -2x_1 + 6x_2 &= h \end{aligned}$$

Solution: Reduce to triangular form.

$$\begin{bmatrix} 3 & -9 & 4 \\ -2 & 6 & h \end{bmatrix} \to \begin{bmatrix} 1 & -3 & \frac{4}{3} \\ -2 & 6 & h \end{bmatrix} \to \begin{bmatrix} 1 & -3 & \frac{4}{3} \\ 0 & 0 & h+\frac{8}{3} \end{bmatrix}$$

The second equation is $0x_1 + 0x_2 = h + \frac{8}{3}$. System is consistent only if $h + \frac{8}{3} = 0$ or $h = \frac{-8}{3}$.