RULES FOR MATRIX ADDITION AND MULTIPLICATION BY SCALARS

$$
\begin{aligned}
&\text{(a)}\ (\mathbf{A}+\mathbf{B})+\mathbf{C}=\mathbf{A}+(\mathbf{B}+\mathbf{C})\\
&\text{(b)}\ \mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A}\\
&\text{(c)}\ \mathbf{A}+\mathbf{0}=\mathbf{A}\\
&\text{(d)}\ \mathbf{A}+(-\mathbf{A})=\mathbf{0}\\
&\text{(e)}\ (\alpha+\beta)\mathbf{A}=\alpha\mathbf{A}+\beta\mathbf{A}\\
&\text{(f)}\ \alpha(\mathbf{A}+\mathbf{B})=\alpha\mathbf{A}+\alpha\mathbf{B}
\end{aligned}
\tag{5}
$$

Each of these rules follows directly from the definitions and the corresponding rules for ordinary numbers.

Because of rule (5)(a), there is no need to put parentheses in expressions like $\mathbf{A}+\mathbf{B}+\mathbf{C}$. Note also that definitions (3) and (4) imply that $\mathbf{A}+\mathbf{A}+\mathbf{A}$ is equal to $3\mathbf{A}$.

PROBLEMS FOR SECTION 15.2

1. Construct the matrix $\mathbf{A} = (a_{ij})_{3\times 3}$, where $a_{ii} = 1$ for $i = 1, 2, 3$, and $a_{ij} = 0$ for $i \neq j$.

2. Evaluate $\mathbf{A}+\mathbf{B}$ and $3\mathbf{A}$ when $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & -1 \\ 5 & 2 \end{pmatrix}$.

3. For what values of u and v does $\begin{pmatrix} (1-u)^2 & v^2 & 3 \\ v & 2u & 5 \\ 6 & u & -1 \end{pmatrix} = \begin{pmatrix} 4 & 4 & u \\ v & -3v & u-v \\ 6 & v+5 & -1 \end{pmatrix}$?

4. Evaluate $\mathbf{A}+\mathbf{B}$, $\mathbf{A}-\mathbf{B}$, and $5\mathbf{A}-3\mathbf{B}$ when

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & -1 \\ 2 & 3 & 7 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 1 & -1 & 5 \\ 0 & 1 & 9 \end{pmatrix}$$

15.3 Matrix Multiplication

The matrix operations introduced so far should seem quite natural. The way in which we define matrix multiplication is not so straightforward. An important motivation for this definition is the way it helps certain key manipulations of linear equation systems.

Consider, for example, the following two linear equation systems:

$$
\text{(i)}\quad \begin{aligned} z_1 &= a_{11}y_1 + a_{12}y_2 + a_{13}y_3 \\ z_2 &= a_{21}y_1 + a_{22}y_2 + a_{23}y_3 \end{aligned}
\qquad\qquad
\text{(ii)}\quad \begin{aligned} y_1 &= b_{11}x_1 + b_{12}x_2 \\ y_2 &= b_{21}x_1 + b_{22}x_2 \\ y_3 &= b_{31}x_1 + b_{32}x_2 \end{aligned}
$$

Now define $\mathbf{A} = \begin{pmatrix} 3 & 4 \\ 7 & -2 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, and $\mathbf{b} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$. Then we see that

$$\mathbf{Ax} = \begin{pmatrix} 3 & 4 \\ 7 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3x_1 + 4x_2 \\ 7x_1 - 2x_2 \end{pmatrix}$$

So the original system is equivalent to the matrix equation

$$\mathbf{Ax} = \mathbf{b}$$

Consider the general linear system (15.1.1) with m equations and n unknowns. Suppose we define

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}, \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

So $\mathbf{A}$ is $m \times n$ and $\mathbf{x}$ is $n \times 1$. The matrix product $\mathbf{Ax}$ is then defined and is $m \times 1$. It follows that

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{array} \quad \text{can be written as} \quad \mathbf{Ax} = \mathbf{b}$$

This very concise notation turns out to be extremely useful.

PROBLEMS FOR SECTION 15.3

SM 1. Compute the products **AB** and **BA**, if possible, for the following:

(a) $\mathbf{A} = \begin{pmatrix} 0 & -2 \\ 3 & 1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} -1 & 4 \\ 1 & 5 \end{pmatrix}$ (b) $\mathbf{A} = \begin{pmatrix} 8 & 3 & -2 \\ 1 & 0 & 4 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 2 & -2 \\ 4 & 3 \\ 1 & -5 \end{pmatrix}$

(c) $\mathbf{A} = \begin{pmatrix} 0 \\ -2 \\ 4 \end{pmatrix}$, $\mathbf{B} = (0, \; -2, \; 3)$ (d) $\mathbf{A} = \begin{pmatrix} -1 & 0 \\ 2 & 4 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 3 & 1 \\ -1 & 1 \\ 0 & 2 \end{pmatrix}$

2. Using the matrices

$$\mathbf{A} = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} -2 & 4 \\ 1 & -2 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 2 & 3 \\ 6 & 9 \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}$$

calculate (i) $3\mathbf{A} + 2\mathbf{B} - 2\mathbf{C} + \mathbf{D}$ (ii) $\mathbf{AB}$ (iii) $\mathbf{C}(\mathbf{AB})$.

3. Let $\mathbf{A} = \begin{pmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{pmatrix}$, $\quad \mathbf{B} = \begin{pmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{pmatrix}$, $\quad \mathbf{C} = \begin{pmatrix} 4 & 1 & 2 \\ 0 & 3 & 2 \\ 1 & -2 & 3 \end{pmatrix}$.

Find the matrices $\mathbf{A} + \mathbf{B}$, $\mathbf{A} - \mathbf{B}$, $\mathbf{AB}$, $\mathbf{BA}$, $\mathbf{A}(\mathbf{BC})$, and $(\mathbf{AB})\mathbf{C}$.

4. Write the following systems as matrix equations:

(a) $\begin{aligned} x_1 + x_2 &= 3 \\ 3x_1 + 5x_2 &= 5 \end{aligned}$ (b) $\begin{aligned} x_1 + 2x_2 + x_3 &= 4 \\ x_1 - x_2 + x_3 &= 5 \\ 2x_1 + 3x_2 - x_3 &= 1 \end{aligned}$ (c) $\begin{aligned} 2x_1 - 3x_2 + x_3 &= 0 \\ x_1 + x_2 - x_3 &= 0 \end{aligned}$

(SM) 5. (a) If **A** is an $m \times n$ matrix and **B** is another matrix such that both products **AB** and **BA** are defined, what must be the dimensions of **B**?

(b) Find all matrices **B** that "commute" with $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$ in the sense that $\mathbf{BA} = \mathbf{AB}$.

6. In Example 3, compute **T(Ts)**.

15.4 Rules for Matrix Multiplication

The algebraic rules in Section 15.2 concerning matrix addition and multiplication by a scalar are all natural and easy to verify. Matrix multiplication is a more complicated operation and we must carefully examine what rules apply. We have already noticed that the commutative law $\mathbf{AB} = \mathbf{BA}$ does NOT hold in general. The following three important rules *are* generally valid, however.

If **A**, **B**, and **C** are matrices whose dimensions are such that the given operations are defined, then:

RULES FOR MATRIX MULTIPLICATION

$(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$	**(associative law)**	(1)
$\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$	**(left distributive law)**	(2)
$(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$	**(right distributive law)**	(3)

Note that both the left and right distributive laws are stated here because, unlike for numbers, matrix multiplication is not *commutative*, and so $\mathbf{A}(\mathbf{B} + \mathbf{C}) \neq (\mathbf{B} + \mathbf{C})\mathbf{A}$ in general.

EXAMPLE 1 Verify (1), (2), and (3) for the following matrices:

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 & -1 \\ 3 & 2 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$$

Solution: All operations of multiplication and addition are defined, with

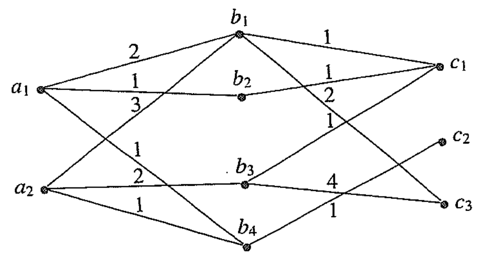

Figure 1

The relevant data can also be represented by the two matrices

$$\mathbf{P}: \begin{matrix} & b_1 & b_2 & b_3 & b_4 \\ a_1 & 2 & 1 & 0 & 1 \\ a_2 & 3 & 0 & 2 & 1 \end{matrix} \qquad \mathbf{Q}: \begin{matrix} & c_1 & c_2 & c_3 \\ b_1 & 1 & 0 & 2 \\ b_2 & 1 & 0 & 0 \\ b_3 & 1 & 0 & 4 \\ b_4 & 0 & 1 & 0 \end{matrix}$$

Each element p_{ij} of the matrix $\mathbf{P}$ represents the number of choices of flight between a_i and b_j, while each element q_{jk} of $\mathbf{Q}$ represents the number of choices of flight between b_j and c_k. How many ways are there of getting from a_i to c_k using two flights, with one connection in country B? Between a_2 and c_3, for example, there are $3 \cdot 2 + 0 \cdot 0 + 2 \cdot 4 + 1 \cdot 0 = 14$ possibilities. This is the inner product of the second row vector in $\mathbf{P}$ and the third column vector in $\mathbf{Q}$. The same reasoning applies for each a_i and c_k. So the total number of flight connections between the different airports in countries A and C is given by the matrix product

$$\mathbf{R} = \mathbf{PQ} = \begin{pmatrix} 2 & 1 & 0 & 1 \\ 3 & 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 1 & 0 & 0 \\ 1 & 0 & 4 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 4 \\ 5 & 1 & 14 \end{pmatrix}$$

PROBLEMS FOR SECTION 15.4

1. Verify the distributive law $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$ when

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 2 & -1 & 1 & 0 \\ 3 & -1 & 2 & 1 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} -1 & 1 & 1 & 2 \\ -2 & 2 & 0 & -1 \end{pmatrix}$$

(SM) 2. Compute the matrix product $(x, y, z) \begin{pmatrix} a & d & e \\ d & b & f \\ e & f & c \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

3. Verify by actual multiplication that $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$ if

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

4. Compute: (a) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 5 & 3 & 1 \\ 2 & 0 & 9 \\ 1 & 3 & 3 \end{pmatrix}$ (b) $(1, 2, -3) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

PROBLEMS FOR SECTION 15.5

1. Find the transposes of $\mathbf{A} = \begin{pmatrix} 3 & 5 & 8 & 3 \\ -1 & 2 & 6 & 2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 2 \end{pmatrix}$, $\mathbf{C} = (1, 5, 0, -1)$.

2. Let $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ -1 & 5 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 0 & 2 \\ 2 & 2 \end{pmatrix}$, and $\alpha = -2$. Compute $\mathbf{A}'$, $\mathbf{B}'$, $(\mathbf{A}+\mathbf{B})'$, $(\alpha\mathbf{A})'$, $\mathbf{AB}$, $(\mathbf{AB})'$, $\mathbf{B}'\mathbf{A}'$, and $\mathbf{A}'\mathbf{B}'$. Then verify all the rules in (2) for these particular values of $\mathbf{A}$, $\mathbf{B}$, and α.

3. Show that $\mathbf{A} = \begin{pmatrix} 3 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 0 & 4 & 8 \\ 4 & 0 & 13 \\ 8 & 13 & 0 \end{pmatrix}$ are symmetric.

4. For what values of a is $\begin{pmatrix} a & a^2-1 & -3 \\ a+1 & 2 & a^2+4 \\ -3 & 4a & -1 \end{pmatrix}$ symmetric?

5. Is the product of two symmetric matrices necessarily symmetric?

SM 6. If $\mathbf{A}_1$, $\mathbf{A}_2$, and $\mathbf{A}_3$ are matrices for which the given products are defined, show that

$$(\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3)' = \mathbf{A}_3'\mathbf{A}_2'\mathbf{A}_1'$$

Generalize to products of n matrices.

7. An $n \times n$ matrix $\mathbf{P}$ is said to be **orthogonal** if $\mathbf{P}'\mathbf{P} = \mathbf{I}_n$.

(a) For $\lambda = \pm 1/\sqrt{2}$, show that $\mathbf{P} = \begin{pmatrix} \lambda & 0 & \lambda \\ \lambda & 0 & -\lambda \\ 0 & 1 & 0 \end{pmatrix}$ is orthogonal.

(b) Show that the 2×2 matrix $\begin{pmatrix} p & -q \\ q & p \end{pmatrix}$ is orthogonal if and only if $p^2 + q^2 = 1$.

(c) Show that the product of two orthogonal $n \times n$ matrices is orthogonal.

SM 8. Define the two matrices $\mathbf{T}$ and $\mathbf{S}$ by $\mathbf{T} = \begin{pmatrix} p & q & 0 \\ \frac{1}{2}p & \frac{1}{2} & \frac{1}{2}q \\ 0 & p & q \end{pmatrix}$, $\mathbf{S} = \begin{pmatrix} p^2 & 2pq & q^2 \\ p^2 & 2pq & q^2 \\ p^2 & 2pq & q^2 \end{pmatrix}$,

and assume that $p + q = 1$.

(a) Prove that $\mathbf{T} \cdot \mathbf{S} = \mathbf{S}$, $\mathbf{T}^2 = \frac{1}{2}\mathbf{T} + \frac{1}{2}\mathbf{S}$, and $\mathbf{T}^3 = \frac{1}{4}\mathbf{T} + \frac{3}{4}\mathbf{S}$.

(b) Conjecture a formula for $\mathbf{T}^n$ $(n = 2, 3, \ldots)$ expressed as a linear combination of $\mathbf{T}$ and $\mathbf{S}$, then prove the formula by induction.

$A' := A^T$

A symmetric ⇕ $A^T = A$

Because 1×1 matrices behave exactly as ordinary numbers with respect to addition and multiplication, we can regard the inner product of **a** and **b** as the matrix product $\mathbf{a}'\mathbf{b}$.

It is usual in economics to regard a typical vector **x** as a column vector, unless otherwise specified. This is especially true if it is a quantity or commodity vector. Another common convention is to regard a price vector as a row vector, often denoted by $\mathbf{p}'$ to suggest that it is the transpose of a column vector. Then $\mathbf{p}'\mathbf{x}$ is the 1×1 matrix, a scalar, that is equal to the inner product $\mathbf{p} \cdot \mathbf{x}$.

Important properties of the inner product follow:

RULES FOR THE INNER PRODUCT

If **a**, **b**, and **c** are n-vectors and α is a scalar, then

(a) $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$

(b) $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$ (2)

(c) $(\alpha\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (\alpha\mathbf{b}) = \alpha(\mathbf{a} \cdot \mathbf{b})$

(d) $\mathbf{a} \cdot \mathbf{a} > 0 \iff \mathbf{a} \neq \mathbf{0}$

Proof: Rules (a) and (c) are easy consequences of the definition.

To prove rule (b), apply the distributive law for matrix multiplication (15.4.2) when **a** is $1 \times n$ whereas **b** and **c** are $n \times 1$.

To prove rule (d), it suffices to note that $\mathbf{a} \cdot \mathbf{a} = a_1^2 + a_2^2 + \cdots + a_n^2$. This is always nonnegative, and is zero only if all the a_i's are 0. ∎

PROBLEMS FOR SECTION 15.7

1. Compute $\mathbf{a} + \mathbf{b}$, $\mathbf{a} - \mathbf{b}$, $2\mathbf{a} + 3\mathbf{b}$, and $-5\mathbf{a} + 2\mathbf{b}$ when $\mathbf{a} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

2. Let $\mathbf{a} = (1, 2, 2)$, $\mathbf{b} = (0, 0, -3)$, and $\mathbf{c} = (-2, 4, -3)$. Find the following:

$$\mathbf{a} + \mathbf{b} + \mathbf{c}, \qquad \mathbf{a} - 2\mathbf{b} + 2\mathbf{c}, \qquad 3\mathbf{a} + 2\mathbf{b} - 3\mathbf{c}$$

(SM) 3. If $3(x, y, z) + 5(-1, 2, 3) = (4, 1, 3)$, find x, y, and z.

4. (a) If $\mathbf{x} + \mathbf{0} = \mathbf{0}$, what do you know about the components of **x**?

   (b) If $0\mathbf{x} = \mathbf{0}$, what do you know about the components of **x**?

(SM) 5. Express the vector $(4, -11)$ as a linear combination of $(2, -1)$ and $(1, 4)$.

6. Solve the vector equation $4\mathbf{x} - 7\mathbf{a} = 2\mathbf{x} + 8\mathbf{b} - \mathbf{a}$ for **x** in terms of **a** and **b**.