LECTURE 3 — FORK 1003

EIVIND ERIKSEN, AUG 11TH 2011 — LINEAR ALGEBRA

SUMMARY:
- INVERSE MATRICES
- SOLVING LINEAR SYSTEMS USING INVERSE MATRICES } [EMEA] 16.6 – 16.7
- CRAMER'S RULE } [EMEA] 16.8

REVIEW:

When A is $n \times n$-matrix, an inverse of A is a matrix A^{-1} such that

$$A \cdot A^{-1} = A^{-1} \cdot A = I_n$$

FACTS:

i) There is an inverse $A^{-1} \iff \det(A) \neq 0$

ii) If $|A| \neq 0$, then A^{-1} is unique

iii) If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is 2×2 and $|A| = ad - bc \neq 0$

then

$$A^{-1} = \frac{1}{ad-bc} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

COMPUTING THE INVERSE:

{ A $n \times n$-matrix with $n \geq 3$. We assume $\det(A) \neq 0$.

① Adjoint formula

The cofactor matrix is the matrix

$$C = \begin{pmatrix} C_{11} & C_{12} & C_{13} & \cdots & C_{1n} \\ C_{21} & C_{22} & C_{23} & \cdots & C_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & C_{n3} & \cdots & C_{nn} \end{pmatrix}$$

with all the cofactors of A.

Recall: $C_{ij} = \underbrace{(-1)^{i+j}}_{\text{sign}} \cdot \underbrace{\left(\begin{array}{l}\text{determinant of matrix you get} \\ \text{from } A \text{ by deleting row } i \text{, column } j\end{array}\right)}_{\text{minor}}$

Ex: $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 5 \end{pmatrix}$

$C_{11} = + \begin{vmatrix} 1 & 3 \\ 0 & 5 \end{vmatrix} = 5 \qquad C_{21} = 0 \qquad C_{31} = 0$

$C_{21} = - \begin{vmatrix} 0 & 1 \\ 0 & 5 \end{vmatrix} = 0 \qquad C_{22} = 5 \qquad C_{32} = 0$

$C_{31} = + \begin{vmatrix} 0 & 1 \\ 1 & 3 \end{vmatrix} = -1 \qquad C_{23} = -3 \qquad C_{33} = 1$

Cofactor matrix: $C = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ -1 & -3 & 1 \end{pmatrix}$

The adjoint matrix of A is $\text{adj}(A) = C^T$

Ex (cont.): $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 5 \end{pmatrix} \Rightarrow C = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ -1 & -3 & 1 \end{pmatrix} \Rightarrow \text{adj}(A) = \begin{pmatrix} 5 & 0 & -1 \\ 0 & 5 & -3 \\ 0 & 0 & 1 \end{pmatrix}$

Formula: When $|A| \neq 0$, we have

$$A^{-1} = \frac{1}{\det(A)} \cdot \text{adj}(A)$$

Ex (cont.): $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 5 \end{pmatrix}$

$\text{adj}(A) = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & -3 \\ 0 & 0 & 1 \end{pmatrix}$

$|A| = 1 \cdot 1 \cdot 5 = 5$

$\Downarrow$

$$A^{-1} = \frac{1}{5} \cdot \begin{pmatrix} 5 & 0 & -1 \\ 0 & 5 & -3 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1/5 \\ 0 & 1 & -3/5 \\ 0 & 0 & 1/5 \end{pmatrix}$$

② Gaussian elimination

$\left\{ \begin{array}{l} A \text{ } n{\times}n\text{-matrix} \\ \text{this method works also when } |A| \neq 0 \end{array} \right.$

SEE [LN] SECTION 2.2

Write down the matrix

$$\left(A \;\middle|\; I_n \right)$$

and use row operations to get a reduced echelon form.

(a) If

$$(A \mid I_n) \rightsquigarrow (I_n \mid B)$$

($\rightsquigarrow$: row operations; $I_n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$; B: something)

then $A^{-1} = B$.

(b) If $(A \mid I_n) \rightsquigarrow (Z \mid *)$

(Z: something else than $I_n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$)

then A^{-1} does not exist.

Ex: $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 5 \end{pmatrix}$ $A^{-1} = ?$

$$\left(\begin{array}{ccc|ccc} ① & 0 & 1 & 1 & 0 & 0 \\ 0 & ① & 3 & 0 & 1 & 0 \\ 0 & 0 & 5 & 0 & 0 & 1 \end{array} \right) :5 \rightarrow \left(\begin{array}{ccc|ccc} ① & 0 & 1 & 1 & 0 & 0 \\ 0 & ① & 3 & 0 & 1 & 0 \\ 0 & 0 & ① & 0 & 0 & 1/5 \end{array} \right) \begin{array}{l} \leftarrow \\ \leftarrow \\ -3, \; -1 \end{array}$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1/5 \\ 0 & 1 & 0 & 0 & 1 & -3/5 \\ 0 & 0 & 1 & 0 & 0 & 1/5 \end{array} \right)$$

Conclusion: since first part is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$,

$$A^{-1} = \begin{pmatrix} 1 & 0 & -1/5 \\ 0 & 1 & -3/5 \\ 0 & 0 & 1/5 \end{pmatrix}$$

Application to linear systems

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n &= b_n \end{aligned} \quad \Rightarrow \quad A \cdot x = \underline{b}$$

linear system ($\#$ equations $= \#$ variables $= n$)

matrix form (A $n \times n$-matrix)

In this case:

$|A| \neq 0$ means one solution

$|A| = 0$ means no solution or infinitely many solutions

If $|A| \neq 0$, we know there is one solution, but how do we find the solution?

① Gaussian elimination (see Lecture 1)

② Matrix algebra (see Lecture 2)

$$A\underline{x} = \underline{b}$$
$$A^{-1} \cdot A\underline{x} = A^{-1}\underline{b}$$
$$x = A^{-1} \cdot \underline{b}$$

A^{-1} exists since $|A| \neq 0$

Ex:
$$\begin{aligned} x \quad\; + z &= 4 \\ y + 3z &= 7 \\ 5z &= 3 \end{aligned}$$
$\Updownarrow$
$$A \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1} \cdot \underline{b} = \begin{pmatrix} 1 & 0 & -1/5 \\ 0 & 1 & -3/5 \\ 0 & 0 & 1/5 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 7 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 - 3/5 \\ 7 - 9/5 \\ 3/5 \end{pmatrix} = \begin{pmatrix} 3.4 \\ 5.2 \\ 0.6 \end{pmatrix}$$

See computation of A^{-1} above

③ Cramer's rule:

Cramer's rule is a way to compute the solution of $A\underline{x} = \underline{b}$ with determinents when A is $n \times n$-matrix with $|A| \neq 0$.

Solution of $A \cdot \underline{x} = \underline{b}$ is

$$x_i = \frac{\det A_i(\underline{b})}{\det A}$$

where $i = 1, 2, 3, \dots, n$ and $A_i(\underline{b})$ is the matrix you get when column i of A is replaced by $\underline{b}$.

Ex:

$$x_1 + x_2 + x_3 = 1$$
$$x_1 - x_2 + x_3 = 4$$
$$x_1 + 2x_2 + 4x_3 = 7$$

$A \cdot \underline{x} = \underline{b}$ with

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 2 & 4 \end{pmatrix} \quad \underline{b} = \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}$$

$$|A| = 1 \cdot (-4-2) - 1 \cdot (4-2) + 1 \cdot (1+1)$$
$$= -6 - 2 + 2 = \underline{-6} \neq 0$$

$\Downarrow$

<u>one</u> solution

first column of A has been replaced by $\underline{b}$.

$$x_1 = \frac{|A_1(\underline{b})|}{|A|} = \frac{\begin{vmatrix} 1 & 1 & 1 \\ 4 & -1 & 1 \\ 7 & 2 & 4 \end{vmatrix}}{-6} = \frac{1 \cdot (-4-2) - 4(4-2) + 7 \cdot (1+1)}{-6} = \frac{-6-8+14}{-6} = \underline{0}$$

$$x_2 = \frac{|A_2(\underline{b})|}{|A|} = \frac{\begin{vmatrix} 1 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 7 & 4 \end{vmatrix}}{-6} = \frac{1 \cdot (16-7) - 1(4-7) + 1(1-4)}{-6} = \frac{9+3-3}{-6} = \underline{-1.5}$$

$$x_3 = \frac{|A_3(\underline{b})|}{|A|} = \frac{\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 4 \\ 1 & 2 & 7 \end{vmatrix}}{-6} = \frac{1 \cdot (-7-8) - 1(7-2) + 1(4+1)}{-6} = \frac{-15 \mp 5 + 5}{-6} = \underline{2.5}$$

<u>Conclusion:</u> $\underline{\underline{x_1 = 0,\ x_2 = -1.5,\ x_3 = 2.5}}$

2.2 The Inverse of a Matrix

The inverse of a real number a is denoted by a^{-1}. For example, $7^{-1} = 1/7$ and

$$7 \cdot 7^{-1} = 7^{-1} \cdot 7 = 1$$

An $n \times n$ matrix A is said to be **invertible** if there is an $n \times n$ matrix C satisfying

$$CA = AC = I_n$$

where I_n is the $n \times n$ identity matrix. We call C the **inverse** of A .

FACT If A is invertible, then the inverse is unique.

Proof: Assume B and C are both inverses of A. Then

$$B = BI = B(\_\_\_\_\_) = (\_\_\_\_\_\_)\_\_\_\_\_\_ = I\_\_\_\_\_\_ = C.$$

So the inverse is unique since any two inverses coincide.■

The inverse of A is usually denoted by A^{-1}.

We have

$$\boxed{AA^{-1} = A^{-1}A = I_n}$$

Not all $n \times n$ matrices are invertible. A matrix which is *not* invertible is sometimes called a **singular** matrix. An invertible matrix is called **nonsingular** matrix.

Theorem 4

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If $ad - bc = 0$, then A is not invertible.

Assume A is any invertible matrix and we wish to solve $A\mathbf{x} = \mathbf{b}$. Then

$$\_\_\_\_\_A\mathbf{x} = \_\_\_\_\_\mathbf{b} \qquad \text{and so}$$

$$I\mathbf{x} = \_\_\_\_\_\_\_\_ \text{ or } \mathbf{x} = \_\_\_\_\_\_\_\_.$$

Suppose $\mathbf{w}$ is also a solution to $A\mathbf{x} = \mathbf{b}$. Then $A\mathbf{w} = \mathbf{b}$ and

$$\_\_\_\_\_A\mathbf{w} = \_\_\_\_\_\mathbf{b} \qquad \text{which means} \qquad \mathbf{w} = A^{-1}\mathbf{b}.$$

So, $\mathbf{w} = A^{-1}\mathbf{b}$, which is in fact the same solution.

We have proved the following result:

Theorem 5

If A is an invertible $n \times n$ matrix, then for each $\mathbf{b}$ in $\mathbf{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

EXAMPLE: Use the inverse of $A = \begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix}$ to solve

$$\begin{aligned} -7x_1 &+ 3x_2 = 2 \\ 5x_1 &- 2x_2 = 1 \end{aligned}.$$

Solution: Matrix form of the linear system:

$$\begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{14-15} \begin{bmatrix} -2 & -3 \\ -5 & -7 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}.$$

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix} \begin{bmatrix} \quad \\ \quad \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$$

Theorem 6 Suppose A and B are invertible. Then the following results hold:

a. A^{-1} is invertible and $(A^{-1})^{-1} = A$ (i.e. A is the inverse of A^{-1}).

b. AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$

c. A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$

Partial proof of part b:

$$(AB)(B^{-1}A^{-1}) = A(\_\_\_\_\_\_\_\_\_\_)A^{-1}$$

$$= A(\_\_\_\_\_\_\_\_\_\_)A^{-1} = \_\_\_\_\_\_\_\_\_\_ = \_\_\_\_\_\_\_\_.$$

Similarly, one can show that $(B^{-1}A^{-1})(AB) = I$.

Theorem 6, part b can be generalized to three or more invertible matrices:

$$(ABC)^{-1} = \_\_\_\_\_\_\_\_\_\_$$

Earlier, we saw a formula for finding the inverse of a 2×2 invertible matrix. How do we find the inverse of an invertible $n \times n$ matrix? To answer this question, we first look at **elementary** matrices.

Elementary Matrices

Definition

An **elementary matrix** is one that is obtained by performing a single elementary row operation on an identity matrix.

EXAMPLE: Let $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$,

$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$ and $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$.

E_1, E_2, and E_3 are elementary matrices. Why?

Observe the following products and describe how these products can be obtained by elementary row operations on A.

$$E_1A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ 2d & 2e & 2f \\ g & h & i \end{bmatrix}$$

$$E_2A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ g & h & i \\ d & e & f \end{bmatrix}$$

$$E_3A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 3a+g & 3b+h & 3c+i \end{bmatrix}$$

If an elementary row operation is performed on an $m \times n$ matrix A, the resulting matrix can be written as EA, where the $m \times m$ matrix E is created by performing the same row operations on I_m.

Elementary matrices are *invertible* because row operations are *reversible*. To determine the inverse of an elementary matrix E, determine the elementary row operation needed to transform E back into I and apply this operation to I to find the inverse.

For example,

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \qquad E_3^{-1} = \begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix}$$

Example: Let $A = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix}$. Then

$$E_1A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$E_2(E_1A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

$$E_3(E_2E_1A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So

$$\boxed{E_3E_2E_1A = I_3}.$$

Then multiplying on the right by A^{-1}, we get

$$E_3E_2E_1A\_\_\_\_\_\_ = I_3\_\_\_\_\_\_.$$

So

$$\boxed{E_3E_2E_1I_3 = A^{-1}}$$

The elementary row operations that row reduce A to $\mathbf{I}_n$ are the same elementary row operations that transform $\mathbf{I}_n$ into $\mathbf{A}^{-1}$.

Theorem 7

An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n, and in this case, any sequence of elementary row operations that reduces A to I_n will also transform I_n to A^{-1}.

Algorithm for finding $\mathbf{A}^{-1}$

Place A and I side-by-side to form an augmented matrix $\begin{bmatrix} A & I \end{bmatrix}$. Then perform row operations on this matrix (which will produce identical operations on A and I). So by Theorem 7:

$$\begin{bmatrix} A & I \end{bmatrix} \text{ will row reduce to } \begin{bmatrix} I & A^{-1} \end{bmatrix}$$

or A is not invertible.

EXAMPLE: Find the inverse of $A = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, if it exists.

Solution:

$$\begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{3}{2} & 1 & 0 \end{bmatrix}$$

So $A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 \\ \frac{3}{2} & 1 & 0 \end{bmatrix}$

Order of multiplication is important!

EXAMPLE Suppose A,B,C, and D are invertible $n \times n$ matrices and $A = B(D - I_n)C$.

Solve for D in terms of A, B, C and D.

Solution:

$$\_\_\_\_\_A\_\_\_\_\_ = \_\_\_\_\_B(D - I_n)C\_\_\_\_\_$$

$$D - I_n = B^{-1}AC^{-1}$$

$$D - I_n + \_\_\_\_ = B^{-1}AC^{-1} + \_\_\_\_$$

$$D = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$