LECTURE 1

FORK1003

EIVIND ERIKSEN, AUG 8TH 2011

LINEAR ALGEBRA

SUMMARY:

- LINEAR SYSTEMS (OF EQUATIONS)
- SOLVING LINEAR SYSTEMS BY GAUSSIAN ELIMINATION

[EMEA] Ch. 15.1, 15.6 + Extra material from Lecture Notes [LN]

- MATRICES, MATRIX ALGEBRA, VECTORS

[EMEA] Ch. 15.2–15.5, 15.7–15.9

LINEAR SYSTEMS (OF EQUATIONS):

Ex:
$$\begin{cases} x+y+z=4 \\ x-y+3z=1 \\ 2x+y-z=3 \end{cases} \qquad \begin{cases} x+y-z-w=1 \\ x \quad +z \quad =4 \end{cases}$$

3 equations
3 variables x, y, z
$\Downarrow$
3×3 linear system

2 equations
4 variables x, y, z, w
$\Downarrow$
2×4 linear system

Defn: A linear system with m equations in the n variables $x_1, x_2, \ldots, x_n$ is of the form

$$\boxed{\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad & \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}} \leftarrow \begin{array}{l} m\times n \\ \text{linear} \\ \text{system} \end{array}$$

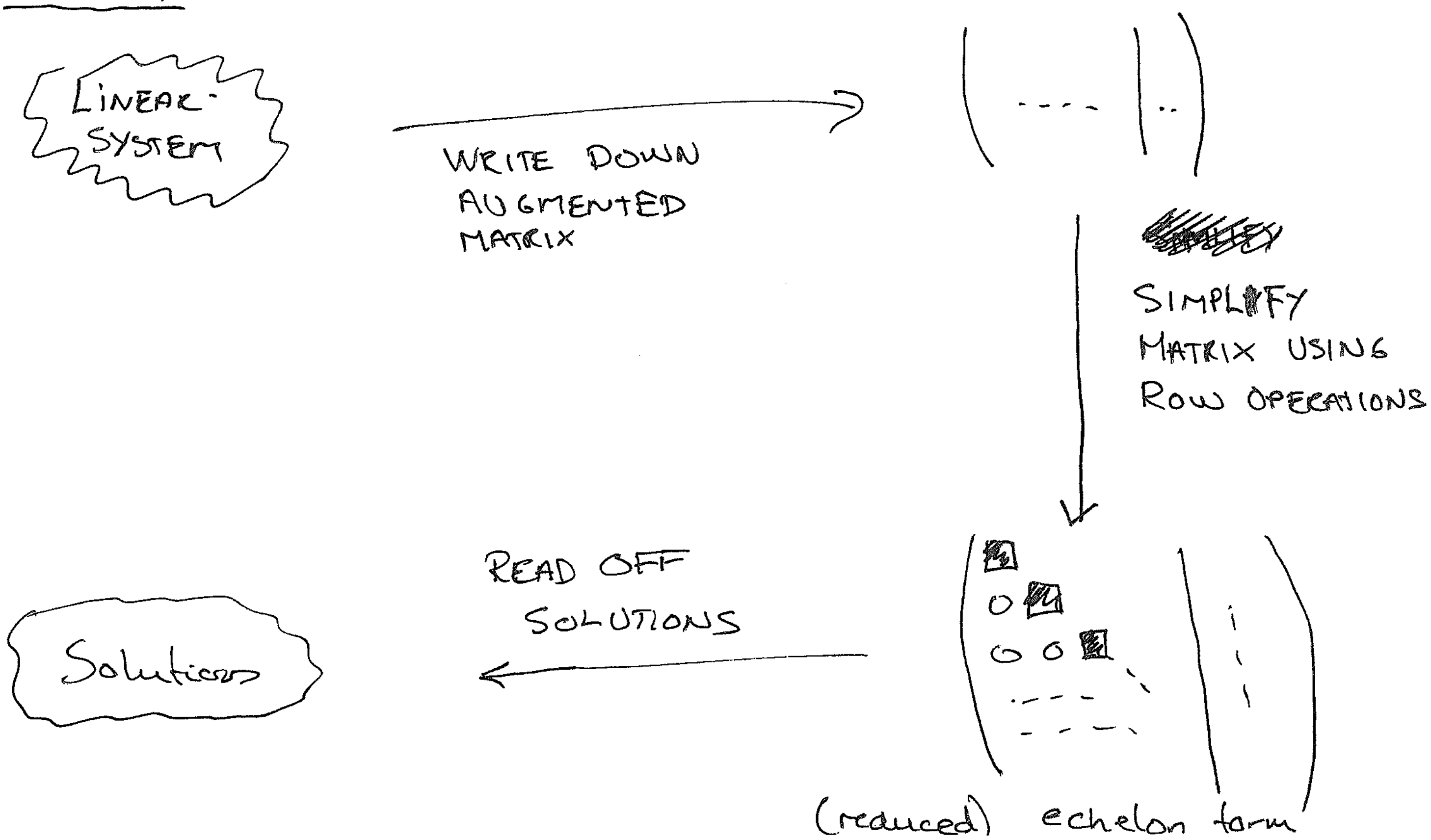
where $a_{11}, a_{12}, \ldots, a_{mn}$ and $b_1, b_2, \ldots, b_m$ are given numbers.

GAUSSIAN ELIMINATION: A METHOD FOR SOLVING ANY LINEAR SYSTEM

- efficient method, used by computers to solve large linear systems
- good also for solving relatively small systems by hand → <u>instructional value</u> for the students; experience show that students who have solved at least a couple of systems that are 3×3 or larger, will gain a much better understanding of linear algebra.

SEE [LN] Section 1.1 - 1.2 for Gaussian elimination

SUMMARY:



Section 1.1: Systems of Linear Equations

A **linear equation:** (in $x_1, \ldots, x_n$)

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

EXAMPLE:

$$4x_1 - 5x_2 + 2 = x_1 \quad \text{and} \quad x_2 = 2(\sqrt{6} - x_1) + x_3$$

$\downarrow$ rearranged $\downarrow$ $\downarrow$ rearranged $\downarrow$

$$3x_1 - 5x_2 = -2 \qquad 2x_1 + x_2 - x_3 = 2\sqrt{6}$$

Not linear:

degree 2   Square root

$$4x_1 - 6x_2 = x_1x_2 \quad \text{and} \quad x_2 = 2\sqrt{x_1} - 7$$

A **system of linear equations** (or a **linear system**):

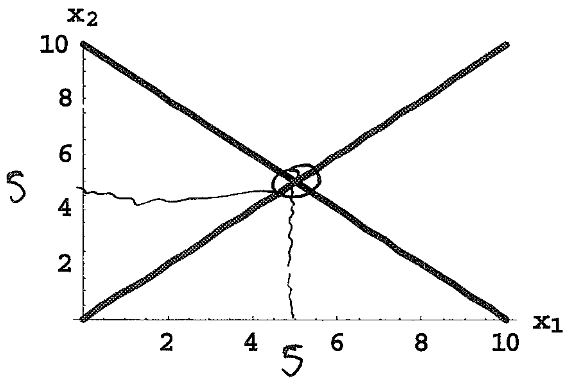
A collection of one or more linear equations involving the same set of variables, say, $x_1, x_2, \ldots, x_n$.

A **solution** of a linear system:

A list $(s_1, s_2, \ldots, s_n)$ of numbers that makes each equation in the system true when the values $s_1, s_2, \ldots, s_n$ are substituted for $x_1, x_2, \ldots, x_n$, respectively.

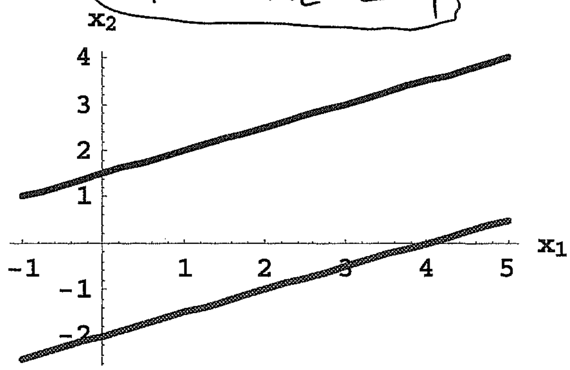
EXAMPLE Two equations in two variables:

$$\begin{aligned} x_1 + x_2 &= 10 \\ -x_1 + x_2 &= 0 \end{aligned}$$



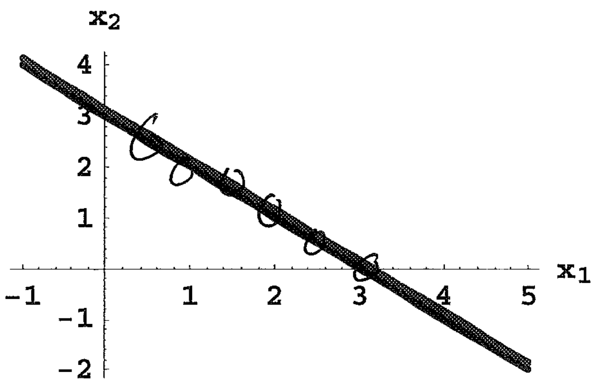
one unique solution

$$\begin{aligned} x_1 - 2x_2 &= -3 \\ 2x_1 - 4x_2 &= 8 \quad : 2 \\ x_1 - 2x_2 &= 4 \end{aligned}$$



no solution

$$\begin{aligned} x_1 + x_2 &= 3 \\ -2x_1 - 2x_2 &= -6 \quad : (-2) \rightarrow x_1 + x_2 = 3 \end{aligned}$$



infinitely many solutions = all points on the (double) line

BASIC FACT: A system of linear equations has either

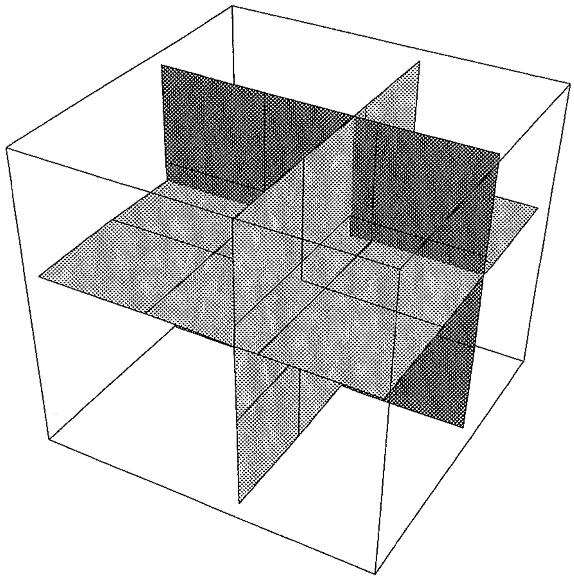
(i) exactly one solution *(consistent)* or

(ii) infinitely many solutions *(consistent)* or

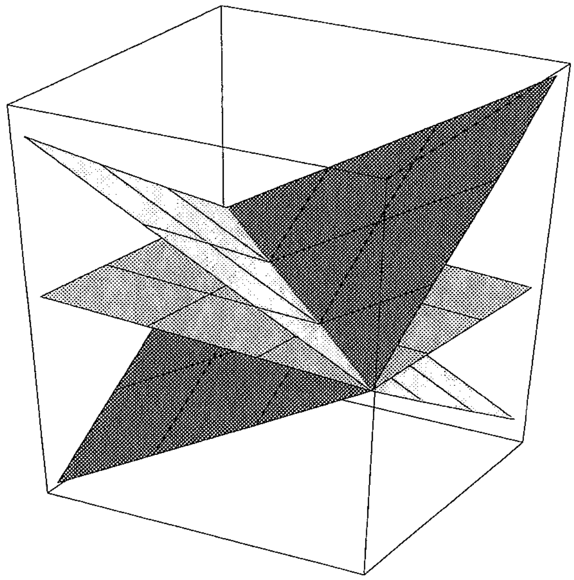
(iii) no solution *(inconsistent)*.

EXAMPLE: Three equations in three variables. Each equation determines a plane in 3-space.

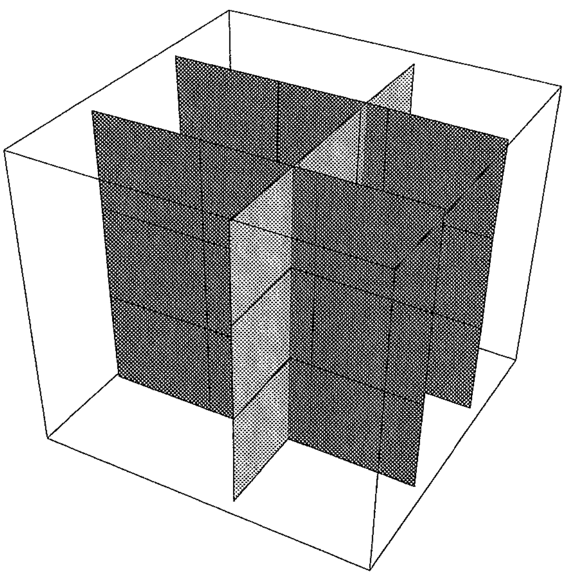
i) The planes intersect in one point. *(one solution)*



ii) The planes intersect in one line. *(infinitely many solutions)*



iii) There is not point in common to all three planes. *(no solution)*



The **solution set:**

- The set of all possible solutions of a linear system.

Equivalent systems:

- Two linear systems with the same solution set.

STRATEGY FOR SOLVING A SYSTEM:

- *Replace one system with an equivalent system that is easier to solve.*

EXAMPLE:

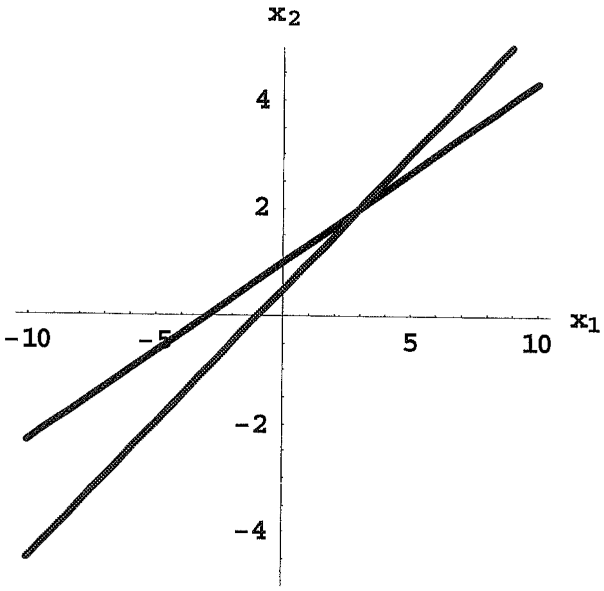
$$\begin{array}{lrcrcr} \underline{\mathrm{I}} & x_1 & - & 2x_2 & = & -1 \\ \underline{\mathrm{II}} & -x_1 & + & 3x_2 & = & 3 \end{array}$$

Added the first row to the second row

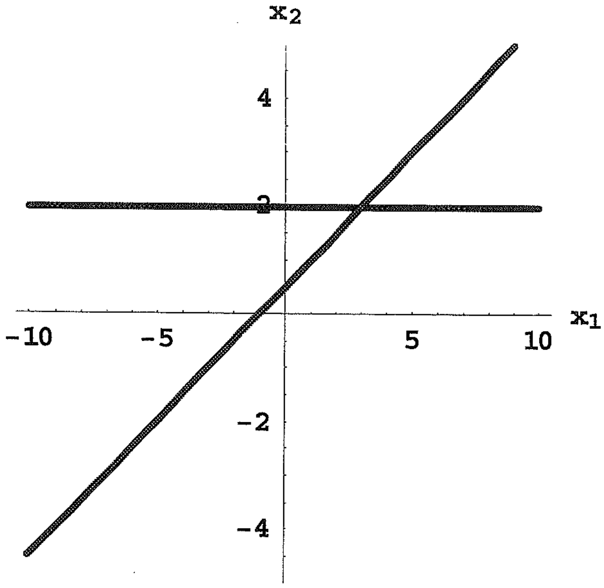
$$\begin{array}{rcrcrl} x_1 & - & 2x_2 & = & -1 & \quad \underline{\mathrm{I}} = a) \\ & & x_2 & = & 2 & \quad \underline{\mathrm{I}} + \underline{\mathrm{II}} = b) \end{array}$$

Added 2x (times) the second row to the first row

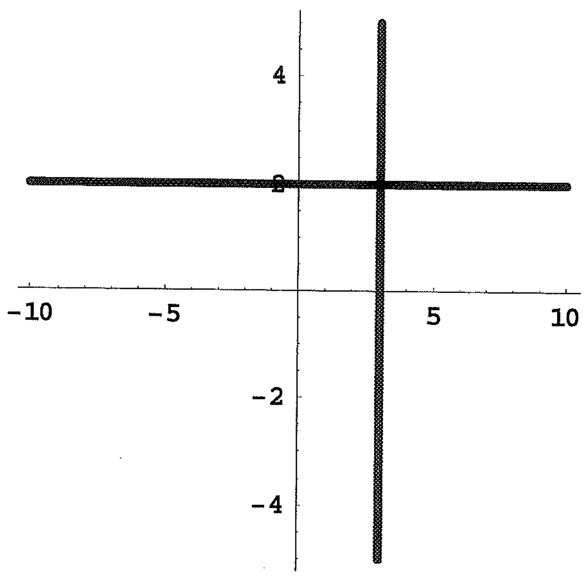
$$\begin{array}{rcrl} x_1 & = & 3 & \quad a) + 2 \cdot b) \\ x_2 & = & 2 & \quad b) \end{array}$$



$$
\begin{aligned}
x_1 - 2x_2 &= -1 \\
-x_1 + 3x_2 &= 3
\end{aligned}
$$



$$
\begin{aligned}
x_1 - 2x_2 &= -1 \\
x_2 &= 2
\end{aligned}
$$



$$
\begin{aligned}
x_1 \phantom{{}+x_2} &= 3 \\
x_2 &= 2
\end{aligned}
$$

Matrix Notation

$$\begin{array}{rcrcr} x_1 & - & 2x_2 & = & -1 \\ -x_1 & + & 3x_2 & = & 3 \end{array} \qquad \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$$

(coefficient matrix)

$$\begin{array}{rcrcr} x_1 & - & 2x_2 & = & -1 \\ -x_1 & + & 3x_2 & = & 3 \end{array} \qquad \left[\begin{array}{rr|r} 1 & -2 & -1 \\ -1 & 3 & 3 \end{array}\right]$$

1.

$R(2) := R(2) + 1 \cdot R(1)$

(augmented matrix)

$$\downarrow$$

$$\begin{array}{rcrcr} x_1 & - & 2x_2 & = & -1 \\ & & x_2 & = & 2 \end{array} \qquad \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

2.

$$\downarrow$$

$$\begin{array}{rcr} x_1 & = & 3 \\ x_2 & = & 2 \end{array} \qquad \left[\begin{array}{rr|r} 1 & 0 & 3 \\ 0 & 1 & 2 \end{array}\right]$$

$x_1 = 3$

$1 \cdot x_1 + 0 \cdot x_2 = 3$

$0 \cdot x_1 + 1 \cdot x_2 = 2$

$x_2 = 2$

Better formulation:

Add a multiple of one row to another row.

Elementary Row Operations:

1. *(Replacement)* Add one row to a multiple of another row.
2. *(Interchange)* Interchange two rows.
3. *(Scaling)* Multiply all entries in a row by a nonzero constant.

Row equivalent matrices: Two matrices where one matrix can be transformed into the other matrix by a sequence of elementary row operations.

Fact about Row Equivalence: If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.

pivot = first non-zero entry in a given row

pivot

EXAMPLE:

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right]$$

4.

get zero under the first pivot

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ & - & 3x_2 & + & 13x_3 & = & -9 \end{array} \qquad \left[\begin{array}{rrrr} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right]$$

2. get second pivot = 1

pivot

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & x_2 & - & 4x_3 & = & 4 \\ & - & 3x_2 & + & 13x_3 & = & -9 \end{array} \qquad \left[\begin{array}{rrrr} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array}\right]$$

3.

echelon form = "staircase"

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & x_2 & - & 4x_3 & = & 4 \\ & & & & x_3 & = & 3 \end{array} \qquad \left[\begin{array}{rrrr} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

4.

-1.

pivot

reduced echelon form:

- all pivots are 1
- only zeros over and under pivots

$$\begin{array}{rcrcr} x_1 & - & 2x_2 & = & -3 \\ & & x_2 & = & 16 \\ & & x_3 & = & 3 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

2.

reduced echelon form

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$$
\begin{array}{ccccc}
x_1 & & & = & 29 \\
 & x_2 & & = & 16 \\
 & & x_3 & = & 3
\end{array}
\qquad
\begin{array}{c}
\begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\
\left[\begin{array}{ccc|c}
1 & 0 & 0 & 29 \\
0 & 1 & 0 & 16 \\
0 & 0 & 1 & 3
\end{array}\right]
\end{array}
$$

Solution: $(29, 16, 3)$

Check: Is $(29, 16, 3)$ a solution of the *original* system?

$$
\begin{array}{rcrcrcr}
x_1 & - & 2x_2 & + & x_3 & = & 0 \\
 & & 2x_2 & - & 8x_3 & = & 8 \\
-4x_1 & + & 5x_2 & + & 9x_3 & = & -9
\end{array}
$$

$$
\begin{array}{rcl}
(29) - 2(16) + 3 & = 29 - 32 + 3 & = 0 \\
2(16) - 8(3) & = 32 - 24 & = 8 \\
-4(29) + 5(16) + 9(3) & = -116 + 80 + 27 & = -9
\end{array}
$$

Two Fundamental Questions (Existence and Uniqueness)

1) Is the system consistent; (i.e. does a solution **exist**?)

2) If a solution exists, is it **unique**? (i.e. is there one & only one solution?)

EXAMPLE: Is this system consistent?

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array}$$

In the last example, this system was reduced to the triangular form:

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & x_2 & - & 4x_3 & = & 4 \\ & & & & x_3 & = & 3 \end{array} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

This is sufficient to see that the system is consistent and unique. Why?

EXAMPLE: Is this system consistent?

$$\begin{aligned} 3x_2 - 6x_3 &= 8 \\ x_1 - 2x_2 + 3x_3 &= -1 \\ 5x_1 - 7x_2 + 9x_3 &= 0 \end{aligned} \qquad \begin{bmatrix} 0 & 3 & -6 & 8 \\ 1 & -2 & 3 & -1 \\ 5 & -7 & 9 & 0 \end{bmatrix}$$

Solution: Row operations produce:

$$\begin{bmatrix} 0 & 3 & -6 & 8 \\ 1 & -2 & 3 & -1 \\ 5 & -7 & 9 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 3 & -1 \\ 0 & 3 & -6 & 8 \\ 0 & 3 & -6 & 5 \end{bmatrix} \rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 3 & -1 \\ 0 & 3 & -6 & 8 \\ 0 & 0 & 0 & -3 \end{array}\right]$$

echelon form

Equation notation of triangular form:

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= -1 \\ 3x_2 - 6x_3 &= 8 \\ 0x_3 &= -3 \end{aligned} \quad \leftarrow \textit{Never true}$$

The original system is inconsistent!

= no solutions

pivot on the right side of | (line) $\Big\}$ $\Longleftrightarrow$ inconsistent (no solutions)

EXAMPLE: For what values of h will the following system be consistent?

$$\begin{aligned} 3x_1 &- 9x_2 &= 4 \\ -2x_1 &+ 6x_2 &= h \end{aligned}$$

Solution: Reduce to triangular form.

$$\begin{bmatrix} 3 & -9 & 4 \\ -2 & 6 & h \end{bmatrix} \to \begin{bmatrix} 1 & -3 & \frac{4}{3} \\ -2 & 6 & h \end{bmatrix} \to \begin{bmatrix} 1 & -3 & \frac{4}{3} \\ 0 & 0 & h+\frac{8}{3} \end{bmatrix}$$

The second equation is $0x_1 + 0x_2 = h + \frac{8}{3}$. System is consistent only if $h + \frac{8}{3} = 0$ or $h = \frac{-8}{3}$.

Section 1.2: Row Reduction and Echelon Forms

leading entry = pivot

Echelon form (or **row echelon form**):

1. All nonzero rows are above any rows of all zeros.
2. Each *leading entry* (i.e. left most nonzero entry) of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zero.

EXAMPLE: Echelon forms

any numbers

pivots

(a) $\begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \\ 0 & 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \end{bmatrix}$

Reduced echelon form: Add the following conditions to conditions 1, 2, and 3 above:

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

EXAMPLE (continued):

Reduced echelon form :

$$\begin{bmatrix} 0 & 1 & * & 0 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 1 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \end{bmatrix}$$

Theorem 1 (Uniqueness of The Reduced Echelon Form):

Each matrix is row-equivalent to one and only one reduced echelon matrix.

Important Terms:

- **pivot position:** a position of a leading entry in an echelon form of the matrix.

- **pivot:** a nonzero number that either is used in a pivot position to create 0's or is changed into a leading 1, which in turn is used to create 0's.

- **pivot column:** a column that contains a pivot position.

(See the Glossary at the back of the textbook.)

EXAMPLE: Row reduce to echelon form and locate the pivot columns.

$$\left[\begin{array}{cccc|c} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{array}\right]$$

Solution

pivot

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix} \quad 1 \quad 2$$

pivot column

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix} \quad -2.5 \quad 1.5$$

Possible Pivots:

would be zero if we find reduced echelon form

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix} \sim \begin{array}{c} \left[\begin{array}{cccc|c} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \end{array} \end{array}$$

echelon form

Original Matrix: $\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$

$\uparrow \quad \uparrow \qquad \uparrow$

pivot columns: 1 2 4

Note: There is no more than one pivot in any row. There is no more than one pivot in any column.

$$\left.\begin{aligned} x_1 + 4x_2 + 5x_3 - 9x_4 &= -7 \\ 2x_2 + 4x_3 - 6x_4 &= -6 \\ -5x_4 &= 0 \end{aligned}\right\}$$

$x_1 + 4(-2x_3 - 3) + 5x_3 = -$

$2x_2 + 4x_3 = -6$

$X_4 = 0$

$x_1 = 3x_3 + 5$

Solve for x_2

$x_2 = \frac{-4x_3 - 6}{2} = \underline{-2x_3 - 3}$

$X_4 = \underline{0}$

Conclusion:

$x_1 = 3x_3 + 5$

$x_2 = -2x_3 - 3$

$x_3 = ?$

$x_4 = 0$

x_1, x_2, x_4 are basic / dependent variables

x_3 is free variable $\Rightarrow$ can be anything

Pivot columns: 1, 2, 4 $\Rightarrow$ x_1, x_2, x_4 dependent

Non-pivot col: 3 $\Rightarrow$ x_3 free

infinitely many solutions

EXAMPLE: Row reduce to echelon form and then to reduced echelon form:

$$\left[\begin{array}{ccccc|c} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{array}\right]$$

Solution:

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix} \sim \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

Cover the top row and look at the remaining two rows for the left-most nonzero column.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} \sim \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

$$\sim \left[\begin{array}{ccccc|c} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right] \text{ (echelon form)}$$

no pivots in last col. $\Rightarrow$ there are sol'ns.

x_1, x_2, x_5 dep.
x_3, x_4 free $\Big\}$ $\Rightarrow$ two degrees of freedom
infinitely many solutions

Final step to create the reduced echelon form:
Beginning with the rightmost leading entry, and working upwards to the left, create zeros above each leading entry and scale rows to transform each leading entry into 1.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 3 & 0 & -6 & 9 & 0 & -72 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$$\sim \begin{array}{c} \\ \\ \\ \text{Pivot col's:} \end{array} \begin{array}{c} \left[\begin{array}{ccccc|c} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right] \\ \begin{array}{ccccc} \uparrow & \uparrow & & & \uparrow \\ 1 & 2 & & & 5 \end{array} \end{array} \Rightarrow x_3, x_4 \text{ free var's}$$

$x_1 = 2x_3 - 3x_4 - 24$

$x_2 = 2x_3 - 2x_4 - 7$

$x_3 = \text{free}$

$x_4 = \text{free}$

$x_5 = 4$

SOLUTIONS OF LINEAR SYSTEMS

= dependent variables

- **basic variable:** any variable that corresponds to a pivot column in the augmented matrix of a system.
- **free variable:** all nonbasic variables.

EXAMPLE:

$$\left[\begin{array}{ccccc|c} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -8 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array}\right]$$

$$\begin{aligned} x_1 + 6x_2 \quad\quad + 3x_4 \quad &= 0 \\ x_3 - 8x_4 \quad &= 5 \\ x_5 &= 7 \end{aligned}$$

pivot columns: 1, 3, 5

basic variables: x_1, x_3, x_5

free variables: x_2, x_4

Final Step in Solving a Consistent Linear System: After the augmented matrix is in **reduced** echelon form and the system is written down as a set of equations:

Solve each equation for the basic variable in terms of the free variables (if any) in the equation.

EXAMPLE:

$$\begin{array}{ccccccc} x_1 & +6x_2 & & +3x_4 & & = 0 \\ & & x_3 & -8x_4 & & = 5 \\ & & & & x_5 & = 7 \end{array} \qquad \begin{cases} x_1 = -6x_2 - 3x_4 \\ x_2 \text{ is free} \\ x_3 = 5 + 8x_4 \\ x_4 \text{ is free} \\ x_5 = 7 \end{cases}$$

(general solution)

The **general solution** of the system provides a parametric description of the solution set. (The free variables act as parameters.) The above system has **infinitely many solutions.**

Why?

Warning: Use only the reduced echelon form to solve a system.

Existence and Uniqueness Questions

EXAMPLE:

$$\left[\begin{array}{cccccc} & 3x_2 & -6x_3 & +6x_4 & +4x_5 & = -5 \\ 3x_1 & -7x_2 & +8x_3 & -5x_4 & +8x_5 & = 9 \\ 3x_1 & -9x_2 & +12x_3 & -9x_4 & +6x_5 & = 15 \end{array}\right]$$

In an earlier example, we obtained the echelon form:

$$\left[\begin{array}{ccccc|c} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right] \quad (x_5 = 4)$$

No equation of the form $0 = c$, where $c \neq 0$, so the system is consistent.

Free variables: x_3 and x_4

Consistent system with free variables	$\Rightarrow$ **infinitely many solutions.**

EXAMPLE:

$$\begin{array}{rrcr} 3x_1 & +4x_2 & = & -3 \\ 2x_1 & +5x_2 & = & 5 \\ -2x_1 & -3x_2 & = & 1 \end{array} \rightarrow \begin{bmatrix} 3 & 4 & -3 \\ 2 & 5 & 5 \\ -2 & -3 & 1 \end{bmatrix}$$

$$\sim \left[\begin{array}{cc|c} 3 & 4 & -3 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right] \longrightarrow \begin{array}{l} 3x_1 + 4x_2 = -3 \\ x_2 = 3 \end{array}$$

Consistent system, no free variables

$\Rightarrow$ **unique solution.**

Theorem 2 (Existence and Uniqueness Theorem)

1. A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column, i.e., if and only if an echelon form of the augmented matrix has no row of the form

$$\begin{bmatrix} 0 & \dots & 0 & b \end{bmatrix} \text{(where } b \text{ is nonzero).}$$

2. If a linear system is consistent, then the solution contains either
(i) a unique solution (when there are no free variables) or
(ii) infinitely many solutions (when there is at least one free variable).

Using Row Reduction to Solve Linear Systems

1. Write the augmented matrix of the system.
2. Use the row reduction algorithm to obtain an equivalent augmented matrix in echelon form. Decide whether the system is consistent. If not, stop; otherwise go to the next step.
3. Continue row reduction to obtain the reduced echelon form.
4. Write the system of equations corresponding to the matrix obtained in step 3.
5. State the solution by expressing each basic variable in terms of the free variables and declare the free variables.

EXAMPLE:

a) What is the largest possible number of pivots a 4×6 matrix can have? Why?

b) What is the largest possible number of pivots a 6×4 matrix can have? Why?

c) How many solutions does a consistent linear system of 3 equations and 4 unknowns have? Why?

d) Suppose the coefficient matrix corresponding to a linear system is 4×6 and has 3 pivot columns. How many pivot columns does the augmented matrix have if the linear system is inconsistent?

MATRICES AND VECTORS

A <u>matrix</u> is a rectangular table of numbers

$$\left.\begin{matrix} m \text{ rows} \\ n \text{ columns}\end{matrix}\right\} \Rightarrow m \times n \text{ matrix}$$

(rows first, then columns)

A <u>vector</u> is a matrix one column (column vector)

<u>Ex:</u>

$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ $\underline{v} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ $\underline{w} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

2x2 - matrix   2-vector   3-vector

$B = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \end{pmatrix}$

2x3 - matrix

<u>Operations:</u>

1) <u>Addition/subtraction:</u> * $A+B$, $A-B$ can be computed if A and B have the same size
   * computed position by position

<u>Ex:</u> $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1+2 & 2+(-1) \\ 3+0 & 4+3 \end{pmatrix} = \underline{\begin{pmatrix} 3 & 1 \\ 3 & 7 \end{pmatrix}}$

$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} - \begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix} = \underline{\begin{pmatrix} -1 & 3 \\ 3 & 1 \end{pmatrix}}$

2) <u>Scaler multiplication:</u> * scaler = number
   * number x matrix = matrix
   * computed position by position

<u>Ex:</u> $2 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \cdot 1 \\ 2 \cdot 2 \end{pmatrix} = \underline{\begin{pmatrix} 2 \\ 4 \end{pmatrix}}$ $3 \cdot \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \underline{\begin{pmatrix} 3 & 6 \\ 9 & 12 \end{pmatrix}}$

3) Matrix multiplication: $\quad$ * matrix × matrix = matrix

* first step: matrix × vector $\rightsquigarrow$ vector

$A \quad \underline{v} \longrightarrow A \cdot \underline{v}$

(matrix-vector multiplication)

Defn: Matrix-vector multiplication

$$A \cdot \underline{v} = \underbrace{\begin{pmatrix} \underline{a_1} & \underline{a_2} & \cdots & \underline{a_n} \end{pmatrix}}_{A} \cdot \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_{\underline{v}} = x_1 \cdot \underline{a_1} + x_2 \cdot \underline{a_2} + \cdots + x_n \cdot \underline{a_n}$$

(linear combination of vectors)

A	$\cdot$	$\underline{v}$	$=$	$A\underline{v}$
m×n matrix		n-vector		m-vector

Ex:

$$\underset{\substack{\uparrow \\ 2\times 3}}{\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \end{pmatrix}} \cdot \underset{\substack{\uparrow \\ \text{3-vector}}}{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}} = 1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} + 1 \cdot \begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \end{pmatrix} = \underset{\substack{\uparrow \\ \text{2-vector}}}{\underline{\underline{\begin{pmatrix} 4 \\ 0 \end{pmatrix}}}}$$

SEE [LN] 1.3-1.4

1.3 VECTOR EQUATIONS

Key concepts to master: linear combinations of vectors ~~and a spanning set.~~

Vector: A matrix with only one column.

Vectors in $\mathbf{R}^n$ (vectors with n entries):

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

Geometric Description of $\mathbf{R}^2$

Vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is the point (x_1, x_2) in the plane.

$\mathbf{R}^2$ is the set of all points in the plane.

Parallelogram rule for addition of two vectors:

If **u** and **v** in $\mathbf{R}^2$ are represented as points in the plane, then $\mathbf{u}+\mathbf{v}$ corresponds to the fourth vertex of the parallelogram whose other vertices are **0**, **u** and **v**. (Note that $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.)

EXAMPLE: Let $\mathbf{u} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Graphs of $\mathbf{u}, \mathbf{v}$ and $\mathbf{u}+\mathbf{v}$ are given below:

$\vec{u}+\vec{v} = \begin{pmatrix} 1+2 \\ 3+1 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

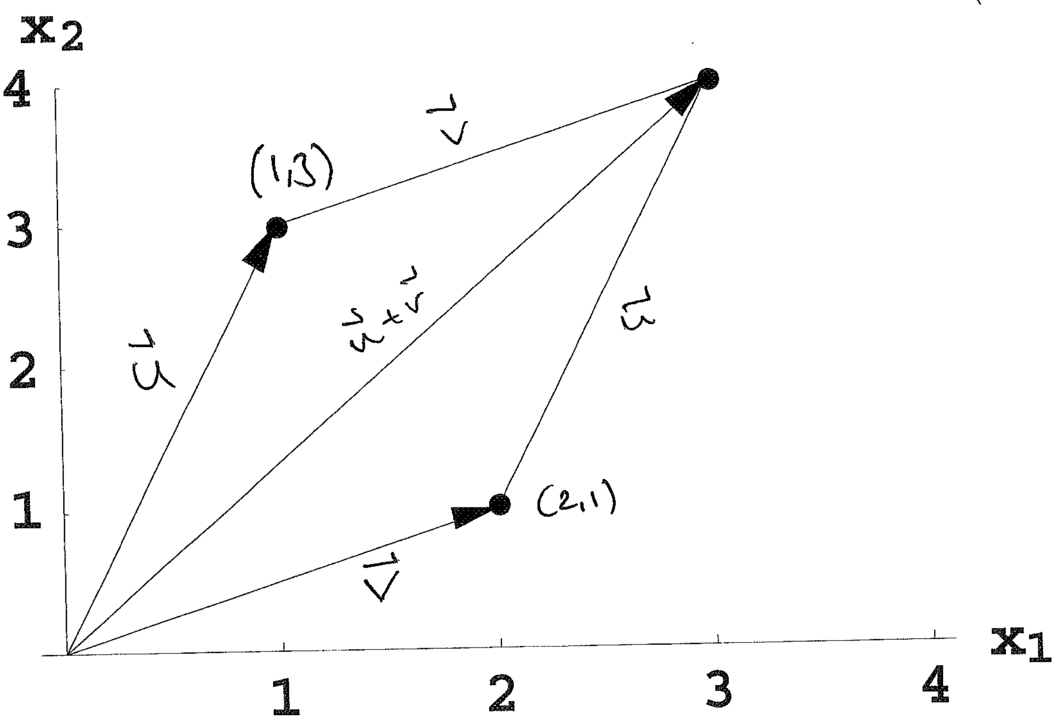
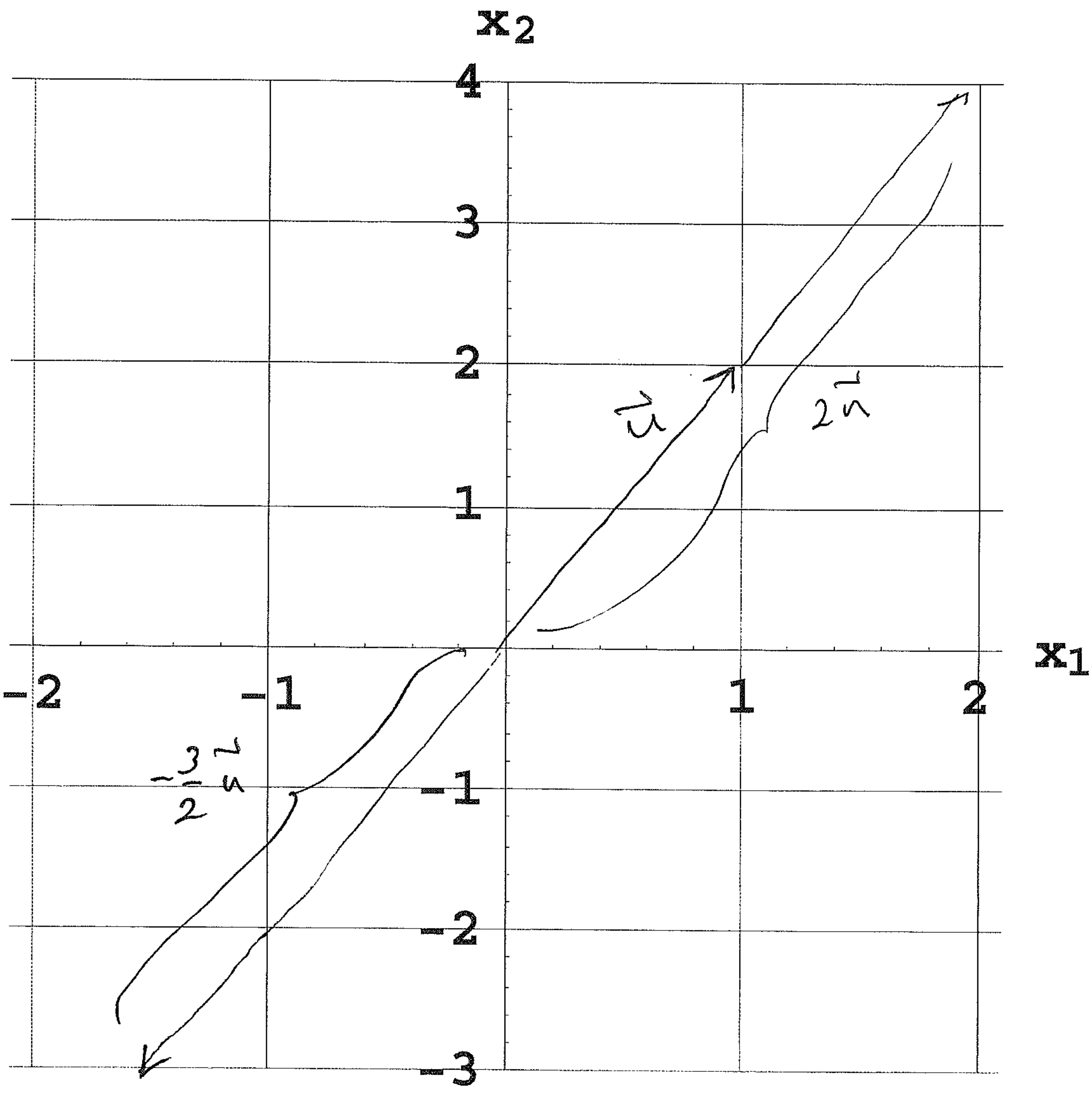


Illustration of the Parallelogram Rule

$2 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

EXAMPLE: Let $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Express $\mathbf{u}$, $2\mathbf{u}$, and $\frac{-3}{2}\mathbf{u}$ on a graph.



Linear Combinations

DEFINITION

Given vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ in $\mathbf{R}^n$ and given scalars $c_1, c_2, \ldots, c_p$, the vector $\mathbf{y}$ defined by

$$\mathbf{y} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p$$

is called a **linear combination** of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ using weights $c_1, c_2, \ldots, c_p$.

Examples of linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$:

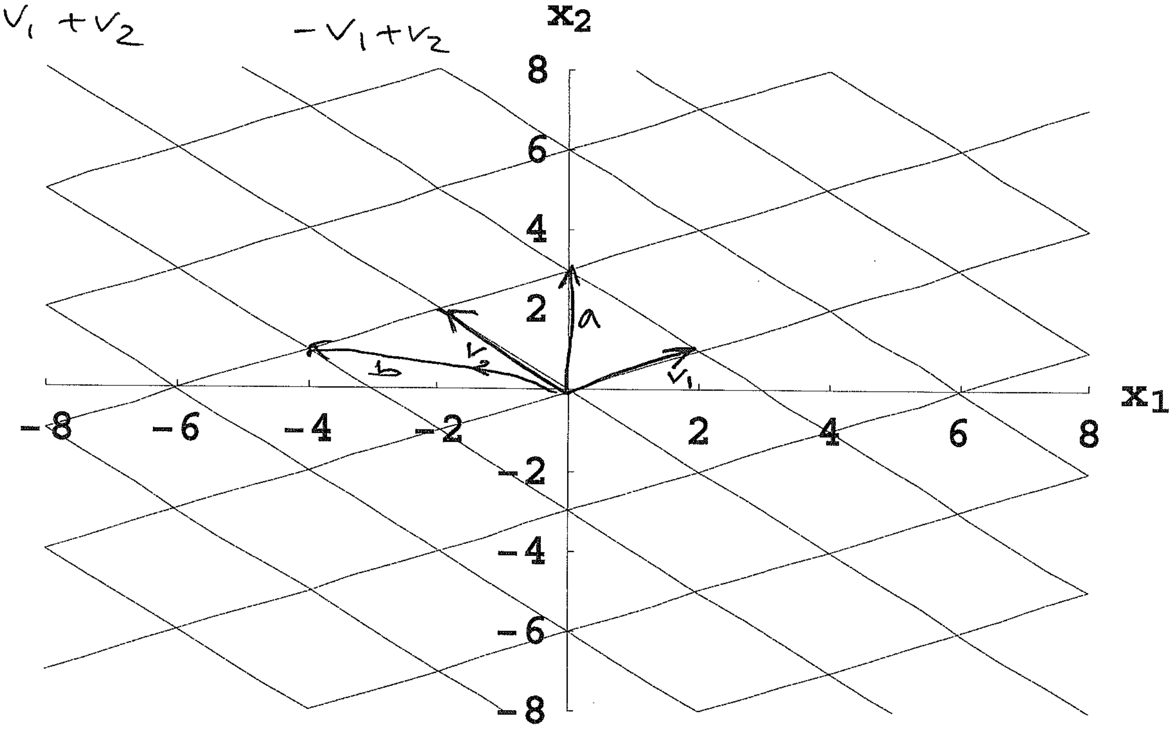
$$3\mathbf{v}_1 + 2\mathbf{v}_2, \qquad \tfrac{1}{3}\mathbf{v}_1, \qquad \mathbf{v}_1 - 2\mathbf{v}_2, \qquad \mathbf{0}$$

$$\tfrac{1}{3}\mathbf{v}_1 = \frac{1}{3} \cdot \underline{v_1} + 0 \cdot \underline{v_2}$$

EXAMPLE: Let $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$. Express each of the following as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$:

$$\mathbf{a} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} 7 \\ -4 \end{bmatrix}$$

$\mathbf{a} = v_1 + v_2$, $\quad \mathbf{b} = -v_1 + v_2$



EXAMPLE: Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 4 \\ 2 \\ 14 \end{bmatrix}$, $\mathbf{a}_3 = \begin{bmatrix} 3 \\ 6 \\ 10 \end{bmatrix}$,

and $\mathbf{b} = \begin{bmatrix} -1 \\ 8 \\ -5 \end{bmatrix}$.

Determine if $\mathbf{b}$ is a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$.

Solution: Vector $\mathbf{b}$ is a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ if can we find weights x_1, x_2, x_3 such that

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b}.$$

Vector Equation (fill-in):

$$x_1 \cdot \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + x_2 \cdot \begin{pmatrix} 4 \\ 2 \\ 14 \end{pmatrix} + x_3 \cdot \begin{pmatrix} 3 \\ 6 \\ 10 \end{pmatrix} = \begin{pmatrix} -1 \\ 8 \\ 5 \end{pmatrix} \Rightarrow \begin{pmatrix} x_1 + 4x_2 + 3x_3 \\ 2x_2 + 6x_3 \\ 3x_1 + 14x_2 + 10x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 8 \\ -5 \end{pmatrix}$$

Corresponding System:

$$\begin{array}{rcrcrcr} x_1 & + & 4x_2 & + & 3x_3 & = & -1 \\ & & 2x_2 & + & 6x_3 & = & 8 \\ 3x_1 & + & 14x_2 & + & 10x_3 & = & -5 \end{array}$$

Corresponding Augmented Matrix:

$$\begin{bmatrix} 1 & 4 & 3 & -1 \\ 0 & 2 & 6 & 8 \\ 3 & 14 & 10 & -5 \end{bmatrix} \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{array}\right] \Rightarrow \begin{array}{l} x_1 = \underline{1} \\ x_2 = \underline{-2} \\ x_3 = \underline{2} \end{array}$$

$$\underline{b} = 1 \cdot \underline{a_1} + (-2) \cdot \underline{a_2} + 2 \cdot \underline{a_3}$$

Review of the last example: $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ and $\mathbf{b}$ are columns of the augmented matrix

$$\begin{array}{c} \begin{bmatrix} 1 & 4 & 3 & -1 \\ 0 & 2 & 6 & 8 \\ 3 & 14 & 10 & -5 \end{bmatrix} \\ \begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{b} \end{array} \end{array}$$

Solution to

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b}$$

is found by solving the linear system whose augmented matrix is

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{b} \end{bmatrix}.$$

A vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

has the same solution set as the linear system whose augmented matrix is

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n & \mathbf{b} \end{bmatrix}.$$

In particular, $\mathbf{b}$ can be generated by a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ if and only if there is a solution to the linear system corresponding to the augmented matrix.

1.4 The Matrix Equation Ax = b

Linear combinations can be viewed as a matrix-vector multiplication.

$$A = \underbrace{\begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix}}_{n} \Big\} m$$

Definition

If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$, and if $\mathbf{x}$ is in $\mathbf{R}^n$, then the **product of A and $\mathbf{x}$**, denoted by $A\mathbf{x}$, is the **linear combination of the columns of A using the corresponding entries in x as weights**. I.e.,

m×n-matrix

n-vector

$$A\mathbf{x} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n$$

matrix product

EXAMPLE:

$$\begin{bmatrix} 1 & -4 \\ 3 & 2 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 7 \\ -6 \end{bmatrix} = 7\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + -6\begin{bmatrix} -4 \\ 2 \\ 5 \end{bmatrix} =$$

$$\begin{bmatrix} 7 \\ 21 \\ 0 \end{bmatrix} + \begin{bmatrix} 24 \\ -12 \\ -30 \end{bmatrix} = \begin{bmatrix} 31 \\ 9 \\ -30 \end{bmatrix}$$

EXAMPLE: Write down the system of equations corresponding to the augmented matrix below and then express the system of equations in vector form and finally in the form $A\mathbf{x} = \mathbf{b}$ where $\mathbf{b}$ is a 3×1 vector.

$$\left[\begin{array}{ccc|c} 2 & 3 & 4 & 9 \\ -3 & 1 & 0 & -2 \end{array}\right]$$

Solution: Corresponding system of equations (fill-in)

$$\left.\begin{aligned} 2x_1 + 3x_2 + 4x_3 &= 9 \\ -3x_1 + x_2 \quad &= -2 \end{aligned}\right\} \Rightarrow x_1 \cdot \begin{pmatrix} 2 \\ -3 \end{pmatrix} + x_2 \cdot \begin{pmatrix} 3 \\ 1 \end{pmatrix} + x_3 \cdot \begin{pmatrix} 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 9 \\ -2 \end{pmatrix}$$

Vector Equation:

$$\underbrace{x_1 \begin{bmatrix} 2 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ 0 \end{bmatrix}}_{} = \begin{bmatrix} 9 \\ -2 \end{bmatrix}.$$

Matrix equation (fill-in):

$$\underbrace{\begin{pmatrix} 2 & 3 & 4 \\ -3 & 1 & 0 \end{pmatrix}}_{= A} \cdot \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{= \underline{x}} = \underbrace{\begin{pmatrix} 9 \\ -2 \end{pmatrix}}_{= \underline{b}}$$

$A \cdot \underline{x} = \underline{b}$.

Matrix form of linear system using matrix-vector multiplication

Three equivalent ways of viewing a linear system:

1. as a system of linear equations;
2. as a vector equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$; or
3. as a matrix equation $A\mathbf{x} = \mathbf{b}$.

THEOREM 3

If A is a $m \times n$ matrix, with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$, and if $\mathbf{b}$ is in $\mathbf{R}^m$, then the matrix equation

$$A\mathbf{x} = \mathbf{b}$$

has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n & \mathbf{b} \end{bmatrix}.$$

Useful Fact:

The equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is a

linear combination of the columns of A.

EXAMPLE: Let $A = \begin{bmatrix} 1 & 4 & 5 \\ -3 & -11 & -14 \\ 2 & 8 & 10 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

Is the equation $A\mathbf{x} = \mathbf{b}$ consistent for all $\mathbf{b}$?

Solution: Augmented matrix corresponding to $A\mathbf{x} = \mathbf{b}$:

$$\begin{bmatrix} 1 & 4 & 5 & b_1 \\ -3 & -11 & -14 & b_2 \\ 2 & 8 & 10 & b_3 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 5 & b_1 \\ 0 & 1 & 1 & 3b_1 + b_2 \\ 0 & 0 & 0 & -2b_1 + b_3 \end{bmatrix}$$

$A\mathbf{x} = \mathbf{b}$ is <u>not</u> consistent for all $\mathbf{b}$ since some choices of $\mathbf{b}$ make $-2b_1 + b_3$ nonzero.

$b_1 = 1,\ b_3 = 1 \Rightarrow -2b_1 + b_3 = -2 + 1 = -1 \neq 0 \Rightarrow$ pivot $\Rightarrow$ <u>inconsistent</u>