EXAMPLE 3 Use (4) to find Q_1^D and Q_2^D in terms of the parameters when

$$2(b+\beta_1)Q_1^D + bQ_2^D = a - \alpha_1$$
$$bQ_1^D + 2(b+\beta_2)Q_2^D = a - \alpha_2$$

Solution: The determinant of the coefficient matrix is

$$\Delta = \begin{vmatrix} 2(b+\beta_1) & b \\ b & 2(b+\beta_2) \end{vmatrix} = 4(b+\beta_1)(b+\beta_2) - b^2$$

Provided $\Delta \neq 0$, by (4) the solutions for Q_1^D is

$$Q_1^D = \frac{\begin{vmatrix} a-\alpha_1 & b \\ a-\alpha_2 & 2(b+\beta_2) \end{vmatrix}}{\Delta} = \frac{2(b+\beta_2)(a-\alpha_1) - b(a-\alpha_2)}{\Delta}$$

For Q_2^D we find a similar expression.

A Geometric Interpretation

Determinants of order 2 have a nice geometric interpretation, as shown in Fig. 1. If the two vectors are situated as in Fig. 1, then the determinant is equal to the hatched area of the parallelogram. If we interchange the two row vectors in the determinant, then it becomes a negative number whose absolute value is equal to the hatched area.

Figure 2 illustrates why the result claimed in Fig. 1 is true. We want to find area T. Note that $2T_1 + 2T_2 + 2T_3 + T = (a_{11} + a_{21})(a_{12} + a_{22})$, where $T_1 = a_{12}a_{21}$, $T_2 = \frac{1}{2}a_{21}a_{22}$, and $T_3 = \frac{1}{2}a_{11}a_{12}$. Then $T = a_{11}a_{22} - a_{21}a_{12}$, by elementary algebra.

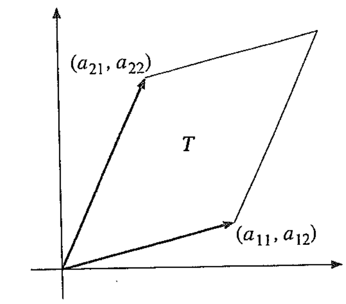

Figure 1 The area T is the absolute value of the determinant $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

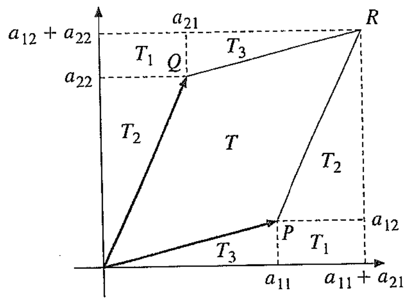

Figure 2 $2T_1 + 2T_2 + 2T_3 + T = (a_{11} + a_{21})(a_{12} + a_{22})$

PROBLEMS FOR SECTION 16.1

1. Calculate the following determinants:

(a) $\begin{vmatrix} 3 & 0 \\ 2 & 6 \end{vmatrix}$ (b) $\begin{vmatrix} a & a \\ b & b \end{vmatrix}$ (c) $\begin{vmatrix} a+b & a-b \\ a-b & a+b \end{vmatrix}$ (d) $\begin{vmatrix} 3^t & 2^t \\ 3^{t-1} & 2^{t-1} \end{vmatrix}$

2. Illustrate the geometric interpretation in Fig. 1 for the determinant in Problem 1(a).

In particular, if we replace a, b, and c by a_{11}, a_{12}, and a_{13}, or by a_{31}, a_{32}, and a_{33}, then the determinant in $(*)$ is 0 because two rows are equal. Hence,

$$a_{11}C_{21} + a_{12}C_{22} + a_{13}C_{23} = 0$$
$$a_{31}C_{21} + a_{32}C_{22} + a_{33}C_{23} = 0$$

That is, the sum of the products of the elements in either row 1 or row 3 multiplied by the cofactors of the elements in row 2 is zero.

Obviously, the argument used in this example can be generalized: If we multiply the elements of any row by the cofactors of a different row, and then add the products, the result is 0. Similarly if we multiply the elements of a column by the cofactors of an alien column, then add.

We summarize all the results in this section in the following theorem:

THEOREM 16.5.1 (COFACTOR EXPANSION OF A DETERMINANT)

Let $\mathbf{A} = (a_{ij})_{n\times n}$. Suppose that the cofactors C_{ij} are defined as in (3). Then:

$$a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} = |\mathbf{A}|$$
$$a_{i1}C_{k1} + a_{i2}C_{k2} + \cdots + a_{in}C_{kn} = 0 \qquad (k \neq i)$$

$$a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} = |\mathbf{A}|$$
$$a_{1j}C_{1k} + a_{2j}C_{2k} + \cdots + a_{nj}C_{nk} = 0 \qquad (k \neq j)$$

Theorem 16.5.1 says that an expansion of a determinant by row i in terms of the cofactors of row k vanishes when $k \neq i$, and is equal to $|\mathbf{A}|$ if $k = i$. Likewise, an expansion by column j in terms of the cofactors of column k vanishes when $k \neq j$, and is equal to $|\mathbf{A}|$ if $k = j$.

PROBLEMS FOR SECTION 16.5

SM 1. Calculate the following determinants:

(a) $\begin{vmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{vmatrix}$ (b) $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & 0 & 11 \\ 2 & -1 & 0 & 3 \\ -2 & 0 & -1 & 3 \end{vmatrix}$ (c) $\begin{vmatrix} 2 & 1 & 3 & 3 \\ 3 & 2 & 1 & 6 \\ 1 & 3 & 0 & 9 \\ 2 & 4 & 1 & 12 \end{vmatrix}$

2. Calculate the following determinants:

(a) $\begin{vmatrix} 0 & 0 & a \\ 0 & b & 0 \\ c & 0 & 0 \end{vmatrix}$ (b) $\begin{vmatrix} 0 & 0 & 0 & a \\ 0 & 0 & b & 0 \\ 0 & c & 0 & 0 \\ d & 0 & 0 & 0 \end{vmatrix}$ (c) $\begin{vmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 3 & 1 & 2 \\ 0 & 4 & 0 & 3 & 4 \\ 6 & 2 & 3 & 1 & 2 \end{vmatrix}$

PROBLEMS FOR SECTION 16.6

1. Prove that the inverse of $\begin{pmatrix} 3 & 0 \\ 2 & -1 \end{pmatrix}$ is $\begin{pmatrix} 1/3 & 0 \\ 2/3 & -1 \end{pmatrix}$.

2. Prove that the inverse of $\begin{pmatrix} 1 & 1 & -3 \\ 2 & 1 & -3 \\ 2 & 2 & 1 \end{pmatrix}$ is $\begin{pmatrix} -1 & 1 & 0 \\ 8/7 & -1 & 3/7 \\ -2/7 & 0 & 1/7 \end{pmatrix}$.

3. Find numbers a and b that make $\mathbf{A}$ the inverse of $\mathbf{B}$ when

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & -1 \\ a & 1/4 & b \\ 1/8 & 1/8 & -1/8 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 6 \\ 1 & 3 & 2 \end{pmatrix}$$

4. Solve the following systems of equations by using Theorem 16.6.2. (See Example 5.)

(a) $\begin{aligned} 2x - 3y &= 3 \\ 3x - 4y &= 5 \end{aligned}$ (b) $\begin{aligned} 2x - 3y &= 8 \\ 3x - 4y &= 11 \end{aligned}$ (c) $\begin{aligned} 2x - 3y &= 0 \\ 3x - 4y &= 0 \end{aligned}$

5. Let $\mathbf{A} = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$. Show that $\mathbf{A}^3 = \mathbf{I}$. Use this to find $\mathbf{A}^{-1}$.

(SM) 6. Given the matrix $\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$.

(a) Calculate $|\mathbf{A}|$, $\mathbf{A}^2$, and $\mathbf{A}^3$. Show that $\mathbf{A}^3 - 2\mathbf{A}^2 + \mathbf{A} - \mathbf{I} = \mathbf{0}$, where $\mathbf{I}$ is the identity matrix of order 3, and $\mathbf{0}$ is the zero matrix.

(b) Show that $\mathbf{A}$ has an inverse and $\mathbf{A}^{-1} = (\mathbf{A} - \mathbf{I})^2$.

(c) Find a matrix $\mathbf{P}$ such that $\mathbf{P}^2 = \mathbf{A}$. Are there other matrices with this property?

7. (a) Let $\mathbf{A} = \begin{pmatrix} 2 & 1 & 4 \\ 0 & -1 & 3 \end{pmatrix}$. Calculate $\mathbf{AA}'$, $|\mathbf{AA}'|$, and $(\mathbf{AA}')^{-1}$.

(b) The matrices $\mathbf{AA}'$ and $(\mathbf{AA}')^{-1}$ in part (a) are both symmetric. Is this a coincidence?

8. (a) If $\mathbf{A}$, $\mathbf{P}$, and $\mathbf{D}$ are square matrices such that $\mathbf{A} = \mathbf{PDP}^{-1}$, show that $\mathbf{A}^2 = \mathbf{PD}^2\mathbf{P}^{-1}$.

(b) Show by induction that $\mathbf{A}^m = \mathbf{PD}^m\mathbf{P}^{-1}$ for any positive integer m.

(SM) 9. Given $\mathbf{B} = \begin{pmatrix} -1/2 & 5 \\ 1/4 & -1/2 \end{pmatrix}$, calculate $\mathbf{B}^2 + \mathbf{B}$, $\mathbf{B}^3 - 2\mathbf{B} + \mathbf{I}$, and then find $\mathbf{B}^{-1}$.

10. Suppose that $\mathbf{X}$ is an $m \times n$ matrix and that $|\mathbf{X}'\mathbf{X}| \neq 0$. Show that the matrix

$$\mathbf{A} = \mathbf{I}_m - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$$

is idempotent—that is, $\mathbf{A}^2 = \mathbf{A}$. (See Problem 15.4.6.)

11. (a) Let $\mathbf{C}$ be an $n \times n$ matrix that satisfies $\mathbf{C}^2 + \mathbf{C} = \mathbf{I}$. Show that $\mathbf{C}^{-1} = \mathbf{I} + \mathbf{C}$.

(b) Show that $\mathbf{C}^3 = -\mathbf{I} + 2\mathbf{C}$ and $\mathbf{C}^4 = 2\mathbf{I} - 3\mathbf{C}$.

We conclude that

$$\mathbf{A}^{-1} = \begin{pmatrix} 7 & -3 & -3 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$$

(Check that $\mathbf{AA}^{-1} = \mathbf{I}$.)

PROBLEMS FOR SECTION 16.7

SM 1. Use Theorem 16.7.1 to calculate the inverses of the following matrices, if they exist:

(a) $\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 0 \\ 0 & 2 & -1 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 0 & 0 \\ -3 & -2 & 1 \\ 4 & -16 & 8 \end{pmatrix}$

2. Find the inverse of $\mathbf{A} = \begin{pmatrix} -2 & 3 & 2 \\ 6 & 0 & 3 \\ 4 & 1 & -1 \end{pmatrix}$.

SM 3. Let $\mathbf{A} = \begin{pmatrix} 0.2 & 0.6 & 0.2 \\ 0 & 0.2 & 0.4 \\ 0.2 & 0.2 & 0 \end{pmatrix}$. Find $(\mathbf{I} - \mathbf{A})^{-1}$.

SM 4. Repeated observations of a phenomenon lead to p different systems of equations

$$\begin{array}{c} a_{11}x_1 + \cdots + a_{1n}x_n = b_{1k} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n = b_{nk} \end{array} \qquad (k = 1, \ldots, p) \qquad (*)$$

which all share the same $n \times n$ coefficient matrix (a_{ij}). Explain how to find the solutions $(x_{k1}, \ldots, x_{kn})$ $(k = 1, \ldots, p)$ of all the systems simultaneously by using row operations to get

$$\begin{pmatrix} a_{11} & \ldots & a_{1n} & b_{11} & \ldots & b_{1p} \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ a_{n1} & \ldots & a_{nn} & b_{n1} & \ldots & b_{np} \end{pmatrix} \sim \begin{pmatrix} 1 & \ldots & 0 & b^*_{11} & \ldots & b^*_{1p} \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & \ldots & 1 & b^*_{1n} & \ldots & b^*_{np} \end{pmatrix}$$

What then is the solution of the system of equations $(*)$ for $k = r$?

SM 5. Use the method in Example 2 to calculate the inverses (provided they exist) of the following matrices (check each result by verifying that $\mathbf{AA}^{-1} = \mathbf{I}$).

(a) $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ (b) $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$ (c) $\mathbf{A} = \begin{pmatrix} 3 & 2 & -1 \\ -1 & 5 & 8 \\ -9 & -6 & 3 \end{pmatrix}$

PROBLEMS FOR SECTION 16.8

SM 1. Use Cramer's rule to solve the following two systems of equations:

(a) $$\begin{aligned} x + 2y - z &= -5 \\ 2x - y + z &= 6 \\ x - y - 3z &= -3 \end{aligned}$$

(b) $$\begin{aligned} x + y \quad\quad &= 3 \\ x \quad + z \quad &= 2 \\ y + z + u &= 6 \\ y \quad + u &= 1 \end{aligned}$$

2. Use Theorem 16.8.1 to prove that the following system of equations has a unique solution for all values of b_1, b_2, b_3, and find the solution.

$$\begin{aligned} 3x_1 + x_2 \quad\quad &= b_1 \\ x_1 - x_2 + 2x_3 &= b_2 \\ 2x_1 + 3x_2 - x_3 &= b_3 \end{aligned}$$

SM 3. Prove that the homogeneous system of equations

$$\begin{aligned} ax + by + cz &= 0 \\ bx + cy + az &= 0 \\ cx + ay + bz &= 0 \end{aligned}$$

has a nontrivial solution if and only if $a^3 + b^3 + c^3 - 3abc = 0$.

16.9 The Leontief Model

In Example 15.1.2 we briefly considered a simple example of the Leontief model. More generally, the Leontief model describes an economy with n interlinked industries, each of which produces a single good using only one process of production. To produce its good, each industry must use inputs from at least some other industries. For example, the steel industry needs goods from the iron mining and coal industries, as well as from many other industries. In addition to supplying its own good to other industries that need it, each industry also faces an external demand for its product from consumers, governments, foreigners, and so on. The amount needed to meet this external demand is called the *final demand*.

Let x_i denote the total number of units of good i that industry i is going to produce in a certain year. Furthermore, let

$$a_{ij} = \text{the number of units of good } i \text{ needed to produce one unit of good } j \tag{1}$$

We assume that input requirements are directly proportional to the amount of the output produced. Then

$$a_{ij}x_j = \text{the number of units of good } i \text{ needed to produce } x_j \text{ units of good } j \tag{2}$$