Obstructions to deforming space curves and non-reduced components of the Hilbert scheme

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§1 Introduction

Obstructions to deforming space curves and ...
### Hilbert scheme

\(V\): a projective variety over \(k = \bar{k}\). \(\text{char } k = 0\)

\(H\): an ample divisor on \(V\).

#### Notation

<table>
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<th>(\text{Hilb} \ V)</th>
<th>= the (full) Hilbert scheme of (V)</th>
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<tr>
<td>(\bigcup \text{ open}) (\text{Hilb}^{\text{sc}} \ V) :</td>
<td>= {smooth connected curves (C \subset V}}</td>
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<tr>
<td>(\text{closed } \bigcup \text{ open}) (\text{Hilb}_{d,g}^{\text{sc}} \ V) :</td>
<td>= {curves of degree degree (d) and genus (g}}</td>
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\((d := (C \cdot H)_V)\)
Hilbert scheme of space curves

\[ V = \mathbb{P}^3: \text{the projective 3-space over } k \]
\[ C \subset \mathbb{P}^3: \text{a closed subscheme of } \text{dim} = 1 \]
\[ d(C): \text{degree of } C \,(= \#(C \cap H)) \]
\[ g(C): \text{genus of } C \, (\text{as a cpt. Riemann surf.}) \]

We study the Hilbert scheme of space curves:

\[ H^{S}_{d,g} := \text{Hilb}^{sc}_{d,g} \mathbb{P}^3 \]
\[ = \left\{ C \subset \mathbb{P}^3 \mid \text{smooth and connected} \right\} \]
\[ \quad d(C) = d \text{ and } g(C) = g \]
Why we study $H^S_{d,g}$?

Some reasons are:

- For every smooth curve $C$, there exists a curve $C' \subset \mathbb{P}^3$ s.t. $C' \simeq C$.
- \[ \text{Hilb}^{sc} \mathbb{P}^3 = \bigsqcup_{d,g} H^S_{d,g} \]
- More recently, the classification of the space curves has been applied to the study of bir. automorphism $\Phi : \mathbb{P}^3 \rightarrow \mathbb{P}^3$.

(for the construction of Sarkisov links [Blanc-Lamy, 2012]).
Some basic facts

- If \( g \leq d - 3 \), then \( H^S_{d,g} \) is irreducible \[\text{[Ein,'86]}\] and \( H^S_{d,g} \) is generically smooth of expected dimension \( 4d \).

- In general, \( H^S_{d,g} \) can become reducible, e.g.
  \[ H^S_{9,10} = W_{1}^{(36)} \sqcup W_{2}^{(36)} \] \[\text{[Noether]}\].

- the Hilbert scheme of arith. Cohen-Macaulay (ACM, for short) curves are smooth \[\text{[Ellingsrud, '75]}\].

\[ C \subset \mathbb{P}^3: \text{ACM} \iff H^1(\mathbb{P}^3, I_C(l)) = 0 \text{ for all } l \in \mathbb{Z} \]

- \( H^S_{d,g} \) can have many generically non-reduced irreducible components, e.g. \[\text{[Mumford’62]}, \text{[Kleppe’87]}, \text{[Ellia’87]}, \text{[Gruson-Peskine’82]}, \text{etc.}\]
Infinitesimal property of the Hilbert scheme

\( V \): a projective variety over \( k \)
\( C \subset V \): a subvariety of \( V \)
\( I_C \): the ideal sheaf defining \( C \) in \( V \)
\( N_{C/V} \): the normal sheaf of \( C \) in \( V \)

**Fact (Tangent space and Obstruction group)**

1. The **tangent space** of \( \text{Hilb} \ V \) at \([C]\) is isomorphic to \( \text{Hom}(I_C, O_C) \cong H^0(C, N_{C/V}) \)
2. Every **obstruction** \( \text{ob} \) to deforming \( C \) in \( V \) is contained in the group \( \text{Ext}^1(I_C, O_C) \). If \( C \) is a locally complete intersection in \( V \), then \( \text{ob} \) is contained in \( H^1(C, N_{C/V}) \)
$W \subset \text{Hilb } V$: an irreducible closed subset of $\text{Hilb } V$.

$[C] \in W$: a closed point of $W$

$C_\eta \in W$: the generic point of $W$

**Definition**

- We say $C$ is unobstructed (resp. obstructed) (in $V$) if $\text{Hilb } V$ is nonsingular (resp. singular) at $[C]$.

- We say $\text{Hilb } V$ is generically smooth (resp. generically non-reduced) along $W$ if $\text{Hilb } V$ is nonsingular (resp. singular) at $C_\eta$. 
Mumford’s example (a pathology)

\[ \text{S} \subseteq \mathbb{P}^3: \text{a smooth cubic surface } (\cong \text{Blow}_{6 \text{ pts}} \mathbb{P}^2) \]

\[ h = S \cap \mathbb{P}^2: \text{a hyperplane section} \]

\[ E: \text{a line on S} \]

There exists a smooth connected curve

\[ C \in |4h + 2E| \subset S \subset \mathbb{P}^3, \]

of degree 14 and genus 24.

Then \( C \) is parametrized by a locally closed subset

\[ W = W^{(56)} \subset H_{14,24}^S \subset \text{Hilb}^{sc} \mathbb{P}^3 \]

of the Hilbert scheme.
The locally closed subset $W^{(56)}$ fits into the diagram

$$\left\{ C \subset \mathbb{P}^3 \left| C \subset \Sigma \text{ (smooth cubic)} \right. \quad \text{and} \quad C \sim 4h + 2E \right\} \longrightarrow =: \ W^{(56)} \subset H^S_{14,24}$$

$$\downarrow \mathbb{P}^{39}\text{-bundle}$$

$$\left( \begin{array}{c} \text{family of smooth} \\ \text{cubic surfaces} \end{array} \right) =: \ U \subset \text{open } |O_{\mathbb{P}^3}(3)| \simeq \mathbb{P}^{19},$$

where we have $\dim |O_S(C)| = 39$ and $h^0(N_{C/\mathbb{P}^3}) = 57.$
\[ H^0(N_{C/P^3}) = \text{the tangent space of } \text{Hilb}^{sc} P^3 \text{ at } [C]. \]
We have the following inequalities:

\[ 56 = \dim W \leq \dim_{[C]} \text{Hilb}^{sc} P^3 \leq h^0(N_{C/P^3}) = 57. \]

Thus we have a dichotomy between (A) and (B):

- **A** \( W \) is an irreducible component of \( (\text{Hilb}^{sc} P^3)_{\text{red}} \).
- **B** \( \text{Hilb}^{sc} P^3 \) is generically non-reduced along \( W \).

- There exists an irreducible component \( W' \supsetneq W \).
- \( \text{Hilb}^{sc} P^3 \) is generically smooth along \( W \).

Which? \( \leadsto \) The answer is (A). (It suffices to prove \( \text{Hilb}^{sc} P^3 \) is singular at the generic point \([C]\) of \( W \). We will see later in §2)
Later many non-reduced components of $\text{Hilb}^{sc} \mathbb{P}^3$ were found by Kleppe[’85], Ellia[’87], Gruson-Peskine[’82], Fløystad[’93] and Nasu[’05]. Moreover, to the question “How bad can the deformation space of an object be?”, Vakil[’06] has answered that

Law (Murphy’s law in algebraic geometry)

Unless there is some a priori reason otherwise, the deformation space may be as bad as possible.
A naive question

Nowadays non-reduced components of Hilbert schemes are not seldom. However,

**Question**

What is/are the most important reason(s) (if any) for their existence?

Our answer is the following: (at least in Mumford’s example,) a \((-1)\)-curve \(E\) (i.e. \(E \simeq \mathbb{P}^1, E^2 = -1\)) on the (cubic) surface is the most important.
A generalization of Mumford’s ex.

Theorem (Mukai-Nasu’09)

\( V \): a smooth projective 3-fold. Suppose that

1. there exists a curve \( E \cong \mathbb{P}^1 \subset V \)
   s.t. \( N_{E/V} \) is generated by global sections,
2. there exists a smooth surface \( S \) s.t. \( E \subset S \subset V \),
   \( (E^2)_S = -1 \) and \( H^1(N_{S/V}) = p_g(S) = 0 \).

Then the Hilbert scheme \( \text{Hilb}^{sc} V \) has infinitely many generically non-reduced components.

In Mumford’s ex., \( V = \mathbb{P}^3 \), \( S \): a smooth cubic, \( E \): a line.
Examples

We have many ex. of generically non-reduced components of $\text{Hilb}^{sc} V$ for uniruled 3-folds $V$.

**Ex.**

1. Let $V$ be a Fano 3-fold and let $-K_V = H + H'$, where $H, H'$: ample. $\exists S \in |H|$ (smooth).

   If $S \neq \mathbb{P}^2$ nor $\mathbb{P}^1 \times \mathbb{P}^1$, then there exists a $(-1)-\mathbb{P}^1 E$ on $S$.

2. Let $V \to F$ be a $\mathbb{P}^1$-bundle over a smooth surface $F$ with $p_g(F) = 0$. Let $S_1$ be a section of $\pi$ and $A$ a sufficiently ample divisor on $F$. Then there exists a smooth surface $S \in |S_1 + \pi^* A|$. Take a fiber $E$ of $S \to F$. 

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§2 Infinitesimal analysis of the Hilbert scheme
In the analysis of Mumford’s ex., we develop some techniques to computing the obstruction to deforming a curve on a uniruled 3-fold (“obstructedness criterion”).

**Setting:**
- $V$: a uniruled 3-fold
- $S$: a surface
- $E$: $(-1)$-$\mathbb{P}^1$ on $S$
- $C$: a curve on $S$ with $C \subset S \subset V$

**Obst. Criterion**

**Non-reduced components of $\text{Hilb}^{sc} V$**
Obstructions and Cup products

\( \tilde{C} \subset V \times \text{Spec } k[t]/(t^2) \): a first order (infinitesimal) deformation of \( C \) in \( V \) (i.e., a tangent vector of \( \text{Hilb } V \) at \([C]\))

\[
\tilde{C} \in \{1\text{st ord. def. of } C\}
\]

\[
\Uparrow\quad \Uparrow \quad \exists 1\text{-to-1}
\]

\[
\alpha \in \text{Hom}(\mathcal{I}_C, \mathcal{O}_C) \quad (\simeq H^0(N_{C/V}))
\]

Define the cup product \( \text{ob}(\alpha) \) by

\[
\text{ob}(\alpha) := \alpha \cup e \cup \alpha \in \text{Ext}^1(\mathcal{I}_C, \mathcal{O}_C),
\]

where \( e \in \text{Ext}^1(\mathcal{O}_C, \mathcal{I}_C) \) is the ext. class of an exact sequence \( 0 \to \mathcal{I}_C \to \mathcal{O}_V \to \mathcal{O}_C \to 0 \).
Fact

A first order deformation $\tilde{C}$ lifts to a deformation over $\text{Spec } k[t]/(t^3)$ if and only if $\text{ob}(\alpha) = 0$.

Remark

- If $\text{ob}(\alpha) \neq 0$, then $\text{Hilb } V$ is singular at $[C]$.
- If $C$ is a loc. complete intersection in $V$, then $\text{ob}(\alpha)$ is contained in the small group $H^1(C, N_{C/V})$ ($\subset \text{Ext}^1(I_C, O_C)$).
Let $C \subset S \subset V$ be a flag of a curve, a surface and a 3-fold (all smooth), and let $\pi_{C/S} : N_{C/V} \to N_{S/V}\big|_C$ be the natural projection.

**Definition**

Define the *exterior component of* $\alpha$ and $\text{ob}(\alpha)$ by

\[
\pi_S(\alpha) := H^0(\pi_{C/S})(\alpha), \\
\text{ob}_S(\alpha) := H^1(\pi_{C/S})(\text{ob}(\alpha)),
\]

respectively.
Let $E \subset S \subset V$ be a flag of a curve, a surface and a 3-fold.

**Definition**

A rational section $\nu$ of $N_{S/V}$ admitting a pole along $E$, i.e.

$$\nu \in H^0(N_{S/V}(E)) \setminus H^0(N_{S/V}),$$

is called an **infinitesimal deformation with a pole**.

**Remark (an interpretation)**

Every inf. def. with a pole induces a 1st ord. def. of the open surface $S^\circ = S \setminus E$ in $V^\circ = V \setminus E$ by the map

$$H^0(N_{S/V}(E)) \rightarrow H^0(N_{S^\circ/V^\circ})$$
Now we are ready to give a sufficient condition for a first order infinitesimal deformation of \( \tilde{C} \) (\( \subset V \times \text{Spec } k[t]/(t^2) \)) of \( C \) in \( V \) to a second order deformation \( \tilde{\tilde{C}} \) (\( \subset V \times \text{Spec } k[t]/(t^3) \)).
We consider $\alpha \in H^0(N_{C/V})$ satisfying the following condition (☆): the ext. comp. $\pi_S(\alpha)$ of $\alpha$ lifts to an inf. def. with a pole along $E$, say $\nu$, and its restriction $\nu|_E$ to $E$ does not belong to the image of the map $\pi_{E/S}(E) := \pi_{E/S} \otimes O_S(E)$.

\[
\begin{array}{ccc}
H^0(N_{C/V}) & \ni \alpha & H^0(N_{E/V(E)}) \\
\downarrow \pi_{C/S} & & \downarrow \pi_{E/S(E)} \\
H^0(N_{S/V}|_C) & \ni \pi_S(\alpha) & H^0(N_{S/V}(E)|_C) \\
\cap & \ni \nu \mapsto \nu|_E & \ni \nu\left|_E \in H^0(N_{S/V}(E)|_E) \right.
\end{array}
\]
Theorem (Mukai-Nasu’09)

Let \( C, E \subset S \subset V \) be as above. Suppose that \( E^2 < 0 \) on \( S \), and let \( \alpha \in H^0(N_{C/V}) \) satisfy (\( \star \)). If moreover,

1. Let \( \Delta := C + K_V|_S - 2E \) on \( S \). Then
   \[
   (\Delta \cdot E)_S = 2(-E^2 + g(E) - 1)
   \] (2.1)

2. the res. map \( H^0(S, \Delta) \to H^0(E, \Delta|_E) \) is surjective, then we have \( \text{ob}_S(\alpha) \neq 0 \).

Remark

If \( E \) is a \((-1)-\mathbb{P}^1\) on \( S \), then the RHS of (2.1) is equal to 0.
How to apply Obstructedness Criterion

(Mumford’s ex. $V = \mathbb{P}^3$)

Every general member $C \subset \mathbb{P}^3$ of Mumford’s ex. $W^{(56)} \subset \text{Hilb}^{sc} \mathbb{P}^3$ is contained in a smooth cubic surface $S$ and $C \sim 4h + 2E$ on $S$ ($E$: a line, $h$: a hyp. sect.).

Let $t_W$ denote the tangent space of $W$ at $[C]$ ($\dim t_W = \dim W = 56$).

Then there exists a first order deformation $\tilde{C} \leftrightarrow \alpha \in H^0(C, N_{C/\mathbb{P}^3}) \setminus t_W$.

of $C$ in $\mathbb{P}^3$.

Claim

$\text{ob}(\alpha) \neq 0$. 
Proof.

Since $H^1(N_{S/\mathbb{P}^3}(E - C)) = 0$, the ext. comp. $\pi_{C/S}(\alpha) \in H^0(N_{S/\mathbb{P}^3}|_C)$ of $\alpha$ has a lifts to a rational section $\nu \in H^0(N_{S/\mathbb{P}^3}(E))$ on $S$ (an inf. def. with a pole). By the key lemma below, the restriction $\nu|_E$ to $E$ is not contained in $\text{im} \pi_{E/S}(E)$. Since $C \sim 4h + 2E = -K_{\mathbb{P}^3}|_S + 2E$, the divisor $\Delta$ is zero. Thus the condition (1) and (2) are both satisfied. \qed

Lemma (Key Lemma)

Since $C$ is general, the finite scheme $Z := C \cap E$ of length 2 is not cut out by any conic in $|h - E| \simeq \mathbb{P}^1$ on $S$. 
§3 Application to Kleppe’s conjecture
Minimal degree $s(W)$ for $W \subset H_{d,g}^S$

\[ \text{Hilb}^{sc} \mathbb{P}^3: \text{the Hilb. sch. of sm. con. curves } C \subset \mathbb{P}^3 \]
\[ H_{d,g}^S \subset \text{Hilb}^{sc} \mathbb{P}^3: \text{the subsch. of curves of deg. } d \text{ and gen. } g \]
\[ W \subset H_{d,g}^S: \text{an irreducible closed subset} \]
\[ C \in W: \text{a general member of } W \]

**Definition (minimal degree of } W) \]

\[ s(W) := \min \left\{ s \in \mathbb{N} \mid H^0(\mathbb{P}^3, \mathcal{I}_C(s)) \neq 0 \right\} \]
\[ = \text{the minimal degree of a surface } S \supset C \]
s-maximal subsets of $H^S_{d,g}$

**Definition (Kleppe’87)**

$W \subset H^S_{d,g}$ is called $s(W)$-maximal if it is maximal w.r.t $s(W)$.

$W$: s-maximal $\implies s(V) > s$ for any closed irreducible subset $V \subsetneq W$.

**Ex. (Mumford’s ex.)**

\[ W = \left\{ C \subset \mathbb{P}^3 \mid C \subset 3S \text{ (sm. cubic)} \text{ and } C \sim 4h + 2E \right\}^\sim \]

is a 3-maximal subset of $H^S_{14,24}$.
First properties of $s$-maximal subsets

In what follows, we assume that

1. $W \subset H^S_{d,g}$ : a $s$-maximal subset
2. a general member $C \subset S$, where $S$: a smooth surface of deg $s$.

**Proposition**

Suppose that $s \leq 4$ and $d > s^2$. Then

1. If $C$ is not a c.i. when $s = 4$, then
   \[ \dim W = (4 - s)d + g + \binom{s+3}{s} - 2 \]
2. If $H^1(\mathbb{P}^3, I_C(s)) = 0$ and if $C$ is not a c.i. when $s = 4$, then $W$ is a generically smooth component of $H^S_{d,g}$.
Let $s = 3$. If $d > 3^2 = 9$, then $\dim W = d + g + 18$.

**Fact**

Every irreducible component of $H_{d,g}^S$ is of dimension greater than or equal to $4d$ ($= \chi(N_C/\mathbb{P}^3)$).

Thus if $W$ is a component, then

$$\dim W \geq 4d \iff g \geq 3d - 18.$$
Conjecture (Kleppe’87, a ver. modified by Ellia)

Let \( d > 9, g \geq 3d - 18 \), and let \( W \subset H^S_{d,g} \) be a 3-maximal subset. If a general member \( C \) of \( W \) satisfies

1. \( H^1(\mathbb{P}^3, \mathcal{I}_C(3)) \neq 0 \), and
2. \( H^1(\mathbb{P}^3, \mathcal{I}_C(1)) = 0 \)

then \( W \) is a **gen. non-reduced** irred. component of \( H^S_{d,g} \).
Remark

1. The linearly normality assumption \((H^1(I_C(1)) = 0)\) was missing in the original ver. of the conjecture. (pointed out by Ellia[’87] with a counterexample).

2. The tangential dimension \(h^0(N_{C/\mathbb{P}^3})\) of \(H^S_{d,g}\) at \([C]\) is greater than \(\text{dim } W\) by \(h^1(I_C(3))\).

3. The subset \(W\) can be described more explicitly in terms of the coordinate \((a; b_1, \ldots, b_6)\) of \(C\) in \(\text{Pic } S \cong \mathbb{Z}^7\)
In the following cases, Kleppe’s conjecture is known to be true.

1. \( g > 7 + \frac{(d - 2)^2}{8} \) and \( d \geq 18 \) [Kleppe’87]

2. \( d \geq 21 \) and \( g > G(d, 5) \) [Ellia’87]

3. \( h^1(\mathbb{P}^3, \mathcal{I}_{C}(3)) = 1 \) [Nasu’05]

---

\(^1\) \( G(d, 5) \) denotes the max. genus of curves of degree \( d \), not contained in any quartic surface. \( G(d, 5) \approx d^2/10 \) for \( d >> 0 \).

\(^2\) proved by computing cup products
Let \( d > 9 \) and \( g \geq 3d - 18 \) and let \( W \subset H^S_{d,g} \) and \( C \) as in Kleppe’s conjecture.

**Lemma**

The following conditions are equivalent:

1. \( C \) is **quadratically normal**, i.e., \( H^1(\mathbb{P}^3, \mathcal{I}_C(2)) = 0 \).
2. \( (C \cdot E) \geq 2 \) for every line \( E \) on \( S \)
3. Let \( C \sim (a; b_1, \ldots, b_6) \) with some basis of \( \text{Pic} \, S \cong \mathbb{Z}^7 \). Then \( b_i \geq 2 \) for all \( i = 1, \ldots, 6 \).
4. Let \( h \in \text{Pic} \, S \) be the cls. of hyp. sections. The base locus of the complete lin. sys. \(|C - 3h|\) contains no double lines \( 2E_i \), and no triple lines \( 3E_i \).
Main Theorem

Theorem (—’09)

Kleppe’s conjecture is true if \( C \) is \textit{quadratically normal}, i.e.,

\[
H^1(\mathbb{P}^3, I_C(2)) = 0.
\]
How to prove Main Theorem

As we see in the Mumford’s ex., it suffices to prove that $\text{ob} (\alpha) \neq 0$ for every

$$\alpha \in H^0(C, N_{C/\mathbb{P}^3}) \setminus t_W,$$

where $t_W$ is the tangent space of $W$ at $[C]$. Note that the ext. comp. $\pi_S(\alpha)$ of $\alpha$ lifts to a rational section

$$v \in H^0(S, N_{S/\mathbb{P}^3}(F)) \setminus H^0(S, N_{S/\mathbb{P}^3}),$$

where $F$ is the fixed component of the lin. sys. $|C - 3h|$. Then we apply the obstructedness criterion for a first order deformation $\tilde{C} \leftrightarrow \alpha$ of $C$ in $\mathbb{P}^3$. 
Another progress has been made:

**Theorem (Kleppe’12)**

Kleppe’s conjecture is true provided that:

1. $b_6 = 2$, $b_5 \geq 4$, $d \geq 21$ and 
   $(a; b_1, \ldots, b_6) \neq (\lambda + 12, \lambda + 4, 4, \ldots, 4, 2), \forall \lambda \geq 2,$

2. $b_6 = 1$, $b_5 \geq 6$, $d \geq 35$ and 
   $(a; b_1, \ldots, b_6) \neq (\lambda + 18, \lambda + 6, 6, \ldots, 6, 1), \forall \lambda \geq 2,$

3. $b_6 = 1$, $b_5 = 5$, $b_4 \geq 7$, $d \geq 35$ and 
   $(a; b_1, \ldots, b_6) \neq (\lambda + 21, \lambda + 7, 7, \ldots, 7, 5, 1), \forall \lambda \geq 2.$
§4 Obstruction to deforming curves on a quartic surface
Quartic surfaces containing a line

Similarly, we can compute the obstructions to deforming curves on a smooth quartic surface.

Assume that:
- $S \subset \mathbb{P}^3$: a smooth quartic surface (a K3 surface),
- $E$: a line on $S$,
- $F (\sim h - E)$: a plane cubic curve cut out by a plane $H \supset E$,
- $\text{Pic } S \cong \mathbb{Z}E \oplus \mathbb{Z}F$, and the intersection matrix is given by

$$
\begin{pmatrix}
E^2 & E \cdot F \\
E \cdot F & F^2
\end{pmatrix} = 
\begin{pmatrix}
-2 & 3 \\
3 & 0
\end{pmatrix}.
$$
Then every curve $C$ on $S$ is expressed in $\text{Pic} S$ by

$$C \sim aE + bF \quad (a, b \geq 0)$$

with $d = a + 3b$ and $g = 3ab - b^2 + 1$.

Let $W$ be a 4-maximal subset containing $[C]$. If $d > 16$ and $a \neq b$, then

$$\dim W = g + 33.$$  

Moreover, we see that....
Theorem (Kleppe’12 (with Ottem))

Suppose that \( d > 16 \) and \( 4 < a < b \). Then

1. If \( 3b - 2a \geq 3 \), then \( h^1(\mathcal{I}_C(4)) = 0 \). In particular, \( W \) is a generically smooth component of \( H^S_{d,g} \).

2. If \( 3b - 2a \leq 2 \), then \( h^1(\mathcal{I}_C(4)) \neq 0 \). Moreover, \( W \) is a generically non-reduced component of \( H^S_{d,g} \).

In fact, we see that

\[
h^1(\mathbb{P}^3, \mathcal{I}_C(4)) = \begin{cases} 
1 & (3b - 2a = 2) \\
2 & (3b - 2a = 1) \\
4 & (3b - 2a = 0)
\end{cases}
\]
By computing cup products, we have proved the following:

**Theorem**

Let $C$ and $W$ be as in the thm. If

$$3b - 2a = 2 \ (\Rightarrow h^1(\mathbb{P}^3, I_C(4)) = 1),$$

then there exists a first order deformation $\tilde{C}$ of $C$ in $\mathbb{P}^3$ which does not lift to a deformation over $\text{Spec } k[t]/(t^3)$.

However, for the other case (where $3b - 2a = 1, 0$) we have not yet proved the obstructedness of a general $C \in W$. 
Quartic surfaces containing a conic

We have many variations of a smooth quartic surface $S$ containing $E \cong \mathbb{P}^1$, e.g., the one containing conics $E_1, E_2$.

Assume that:

- $S \subset \mathbb{P}^3$: a smooth quartic surface (a K3 surface),
- $E_1, E_2$: conics on $S$ such that $h \sim E_1 + E_2$,
- $\text{Pic } S \cong \mathbb{Z}E_1 \oplus \mathbb{Z}E_2$, and the intersection matrix is given by

$$
\begin{pmatrix}
E_1^2 & E_1 \cdot E_2 \\
E_1 \cdot E_2 & E_2^2
\end{pmatrix}
= 
\begin{pmatrix}
-2 & 4 \\
4 & -2
\end{pmatrix}
$$

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Then every curve $C$ on $S$ is expressed in $\text{Pic} \ S$ by

$$C \sim aE + bF \quad (a, b \geq 0)$$

with $d = 2a + 2b$ and $g = 4ab - a^2 - b^2 + 1$. Let $W$ be a 4-maximal subset containing $[C]$. If $d > 16$ and $a \neq b$, then

$$\dim W = g + 33.$$ 

Moreover, we see that....
Theorem (Kleppe’12 (with Ottem))

Suppose that \( d > 16 \) and \( a \neq b > 4 \). If

\[
\frac{b + 4}{2} \leq a \leq 2b - 4,
\]

then \( h^1(\mathbb{P}^3, I_C(4)) = 0 \). In particular, \( W \) is a generically smooth component of \( H^S_{d,g} \).

Otherwise, we see that

\[
h^1(\mathbb{P}^3, I_C(4)) = \begin{cases} 
1 & (2b - a = 3) \\
4 & (2b - a = 2) \\
9 & (2b - a = 1) \\
16 & (2b - a = 0)
\end{cases}
\]
By computing cup products, we have proved the following:

**Theorem**

Let $C$ be a general member of $W$, and suppose that

$$2b - a = 3 \quad \Rightarrow \quad h^1(\mathbb{P}^3, \mathcal{I}_C(4)) = 1.$$

Then there exists a first order deformation $\tilde{C}$ of $C$ in $\mathbb{P}^3$ which does not lift to a deformation over $\text{Spec } k[t]/(t^3)$.

However, for the other case (where $2b - a = 2, 1, 0$) we have not yet proved the obstructedness of a general $C \in W$. 
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